

STRONG REACTIONS IN QUANTUM SUPER PDE'S.I: QUANTUM HYPERCOMPLEX EXOTIC SUPER PDE'S

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ABSTRACT. In order to encode strong reactions of the high energy physics, by means of quantum nonlinear propagators in the Prástaro's geometric theory of quantum super PDE's, some related geometric structures are further developed and characterized. In particular *super-bundles of geometric objects* in the category \mathfrak{Q}_S of quantum supermanifolds are considered and quantum Lie derivative of sections of super bundle of geometric objects are calculated. Quantum supermanifolds with classic limit are classified with respect to the holonomy groups of these last commutative manifolds. A theorem characterizing quantum super manifolds with structured classic limit as super bundles of geometric objects is obtained. A theorem on the characterization of *chi*-flow on suitable quantum manifolds is proved. This solves a previous conjecture too. *Quantum instantons* and *quantum solitons* are defined as useful generalizations of the previous ones, well-known in the literature. Quantum conservation laws for quantum super PDEs are characterized. Quantum conservation laws are proved work for evaporating quantum black holes too. Characterization of observed quantum nonlinear propagators, in the observed quantum super Yang-Mills PDE, by means of conservation laws and observed energy is obtained. Some previous results by A. Prástaro about generalized Poincaré conjecture and quantum exotic spheres, are generalized to the category $\mathfrak{Q}_{hyper,S}$ of hypercomplex quantum supermanifolds. (This is the first part of a work divided in three parts. For part II and III see [84, 85].)

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1. Introduction

The algebraic topologic theory of quantum (super) PDE's formulated by A. Prástaro, allows to directly encode quantum phenomena in a category of noncommutative manifolds (quantum (super)manifolds) and to finally solve the problem of unification, at quantum level, of gravity with the fundamental forces [61, 64, 65, 66, 67, 68, 70, 73, 75, 77, 78]. In particular, this theory allowed to recognize the mechanism of mass creation/destruction, as a natural geometric phenomenon related to

the algebraic topologic structure of quantum (super) PDEs encoding the quantum system under study [78].

Let us emphasize that with the algebraic topology of quantum (super) PDE's, formulated by A. Prástaro, we can also go beyond the *paradox of the wave-particle duality*, and the probabilistic interpretation (*Copenhagen interpretation*) of the wave function, introduced in quantum mechanics by De Broglie [9, 10], that even if accepted for practical reasons, was conceptually non well convincing. (Let us recall the Einstein's slogan, "*God does not play dice ...*", claimed in the famous Fifth Solvay International Conference, 1927.) Really in the geometric formulation of quantum PDEs, we solve the wave-particle duality and obtain a purely geometric formulation of quantum phenomena. In a sense the situation is similar to the paradoxical *ether*, as the medium of propagation of electromagnetic radiation, wrongly assumed as a necessary support in the Maxwell's dynamics of electromagnetism. In fact, nowadays we can encode quantum particles, and interactions between them, uniquely as geometric objects (*p-chains*), solutions of suitable quantum (super) PDEs, i.e., PDEs in the category \mathfrak{Q}_S of quantum supermanifolds, as introduced by A. Prástaro. Really it is the same concept of "fundamental particles", related to the belief that there are some *fundamental bricks* building all Universe, in a LEGO blocks game sort, that the noncommutative and nonlinear theory of quantum super PDEs proves to be completely unjustified.¹ From this point of view it is not difficult to understand that also the concept of *confinement* reserved to quarks can be "broken" under suitable energy conditions, by generating, e.g., meson decays into quarks, antiquarks and gluons, or baryon decays into quarks and gluons.² With this respect, it is also clear that the Gell-Mann's *standard model* for hadrons, cannot be considered the last frontier in High Energy Physics !

The geometric theory of quantum (super) PDE's allows us to recover also the positive aspects of classical instantons. In fact, we can define *quantum super-instantons*, singular solutions of the quantum super Yang-Mills equations, that being localized in the quantum Minkowskian spacetime M , justify their names (similarly to classical instantons). Furthermore, singular solutions can encode tunneling effects, that, as it is well known, is a phenomenon that can be associated to classical instantons. Furthermore quantum particles can be seen as *quantum super-solitons* in the framework of solutions of quantum super Yang-Mills equations.

Aim of this work, divided in three parts, is to show how nuclear and subnuclear reactions can be encoded as boundary value problems in the algebraic topology of quantum (super) PDE's, as formulated by A. Prástaro. For this is fundamental the concept of quantum nonlinear propagator, and its representation in terms of elementary reactions. (Part II and Part III are quoted in [84, 85].)³ This aspect

¹Nowadays we have also experimental evidences that such fundamental bricks do not exist. What it is considered "fundamental" can be broken, when enough energy is available and suitable apparatus are arranged for. (See, e.g., electron decay into spinon and orbiton [90], or electrons with fractional electric charges [21], and LHC experiments producing new quasiparticles by breaking proton.)

²A quark-gluon plasma state, where quarks and gluons are not more confined, has been recently observed in some experiments. ("LHC experiments bring new insight into primordial universe" (Press release). CERN. 26 November 2010.)

³For complementary information on High Energy Physics related to the subject considered in both part I and part II, see, e.g., Refs. [5, 7, 14, 15, 16, 17, 19, 20, 21, 22, 25, 26, 35, 40, 41, 53, 55, 56, 88, 96].

is related to a decomposition theorem for quantum nonlinear propagators that we prove for solutions of boundary value problems in quantum super PDEs, and that will be developed in the second and third parts. This first part is instead devoted to some important preparatory subjects on the geometry of quantum super PDEs that will be utilized in the next parts, to encode strong reaction dynamics. In particular, we will generalize to the category \mathfrak{Q}_S of quantum supermanifolds, the *structure of super-bundle of geometric objects*, first introduced by A. Prástaro, in the framework of category of commutative manifolds in [57, 58]. This structure gives a general framework where characterize physical fields in a fully covariant way. In particular it applies to PDEs defined on some higher order G -structures.

Another subject that will be developed in this first part and that is very important to encode strong reactions, is the characterization of conservation laws for PDE's built in the category \mathfrak{Q}_S . These are functions defined on the integral bordism groups of such equations and belonging to suitable Hopf algebras (*full quantum Hopf algebras*). In particular, we specialize our calculations on the quantum super Yang-Mills equations, quantum black holes as particular sections there, and the characterization of observed quantum nonlinear propagators by means of conservation laws and observed quantum energy.

The last subject considered in this first part, is devoted to PDE's in the category $\mathfrak{Q}_{hyper,S}$ of quantum hypercomplex supermanifolds, as defined in [83], and will focus our attention on *quantum exotic super PDE's*, i.e., quantum super PDE's where we can embed *quantum exotic super-spheres*. For such Cauchy data we will generalize our previous results on quantum exotic PDE's [83]. Exotic boundary value problems are of particular interest in strong reactions encoding quantum processes occurring in high energy physics, as we will show in the second and third parts of this work. In the following we show as it is organized the paper and list the main results. 2. The concept of *super bundle of geometric objects* (see [57, 58]) is generalized to the category \mathfrak{Q}_S for quantum super manifolds. This is necessary in order to obtain fully covariant structures, as requested in generalized mathematical structures that aim encode physical structures. Theorem 2.9 that characterizes PDEs as superbundles of geometric objects. Theorem 2.10 classifies quantum supermanifolds, with classic limit, with respect to the holonomy groups of these last commutative manifolds. Theorem 2.12 characterizes quantum super manifolds with structured classic limit as super bundles of geometric objects. Theorem 2.16. Here we define *quantum \hat{Q}^7* any quantum (super)manifold M having as classic limit $M_C = Q^7$. (This is the spinor bundle $Q^7 \rightarrow S^3$, with \mathbb{C}^2 fibers.) We say *quantum χ -flow* on M , any quantum flow that projects on the classic limit in a χ -flow [4]. Then we have that a quantum χ -flow represents diffeomorphically any homotopy 3-sphere, $\Sigma^3 \subset Q^7$, onto $S^3 \subset Q^7$. This also solves a previous conjecture formulated in [4] for commutative manifolds. Definition 2.18 gives useful generalizations of previous, well-known definitions of instantons and solitons in the category \mathfrak{Q}_S . 3. Previous Prástaro's results on the quantum conservation laws of quantum super PDE's are resumed. These are necessary in order to obtain the new results on quantum strong reactions obtained in the second part. Theorem 3.17 characterizes quantum black-hole dynamics by means of quantum integral characteristic supernumbers: these are conserved through a non-weak quantum evaporating black-hole. Theorem 3.20 and Corollary 25 give a precise meaning to the phenomenological concept of conservation of energy during an observed quantum process. 4. Here we explicitly extend to the

category $\mathfrak{Q}_{hyper,S}$ of hypercomplex quantum supermanifolds, previous results, in the category \mathfrak{Q}_{hyper} of hypercomplex quantum manifolds, given in [83] and in the category of quantum supermanifold [70, 75]. In particular, Theorem 4.6 explicitly extends to the category $\mathfrak{Q}_{hyper,S}$ a previous result about the generalized Poincaré conjecture in the category \mathfrak{Q}_S . (See [75].) Theorem 4.7 classifies diffeomorphic classes of hypercomplex quantum homotopy superspheres. Theorem 4.39 classifies integral bordism groups in quantum hypercomplex exotic super PDEs. Theorem 4.40 characterizes integral h-cobordism in quantum hypercomplex Ricci flow super PDEs.

2. Super-bundles of geometric objects in the category \mathfrak{Q}_S

The quantum fundamental fields of physics, i.e., electromagnetic, gravitational and nuclear fields, all must obey the full covariance requirement that can be codified generalizing to the category \mathfrak{Q}_S the *structure of super-bundle of geometric objects*, first introduced by A. Prástaro, in the framework of category of commutative manifolds in [57, 58].⁴

Remark 2.1 (The concept of full covariance in \mathfrak{Q}_S). *Similarly to what happens in the classical field theory, the concept of full covariance is fundamental in any quantum field theory. A quantum field is considered a section s of a suitable fiber bundle $\pi : W \rightarrow M$ in the category \mathfrak{Q}_S . To say that s is fully covariant it means that for any local diffeomorphism ϕ of the base M we can calculate the pull-back ϕ^*s of s by means of ϕ :*

$$\phi^*s \equiv \mathbb{B}(\phi) \circ s \circ \phi^{-1}$$

where $\mathbb{B}(\phi)$ is a local application on W , canonically associated to ϕ on M , such that the following diagram is commutative:

$$\begin{array}{ccc} W|_U & \xrightarrow{\mathbb{B}(\phi)} & W|_{\overline{U}} \\ \pi \downarrow & & \downarrow \pi \\ U & \xrightarrow{\phi} & \overline{U} \end{array}$$

If this circumstance is verified, we say that $\pi : W \rightarrow M$ is a natural fiber bundle (or fiber bundle of geometric objects), in the category \mathfrak{Q}_S . We write: $(\pi : W \rightarrow M, \mathbb{B})$. A generalization of this concept is that of super-fiber bundle of geometric objects introduced in [57, 58]. A quantum physical field is fully covariant iff it is a quantum superfield, that is a section of such a structure.

Definition 2.2. A structure of superbundle of geometric objects in the category \mathfrak{Q}_S , is given by two fiber bundles $W \xrightarrow{\pi_W} M \xleftarrow{\pi_B} B$ over the same base M and a covariant functor $\mathbb{B} : \mathcal{C}(B) \rightarrow \mathcal{C}(W)$, where $\mathcal{C}(B)$, (resp. $\mathcal{C}(W)$) is the category whose objects are open subbundles of B (respectively, open subbundles of W) and whose morphisms are the local fiber bundle automorphisms between those objects such that:

⁴See also Prástaro's algebraic topology of PDEs [59, 60, 61, 62, 63, 67, 69, 71, 74, 76, 79, 80, 81, 82] in order to better understand its generalization to the category of quantum supermanifolds. Interesting applications can be found in [2, 3, 39, 86, 87]. For basic information on the geometry of PDE's see [12, 18, 23, 27, 38]. For basic information on algebraic topology see [34, 42, 43, 44, 46, 49, 54, 91, 92, 93, 94, 97, 98, 99, 100, 101, 102].

TABLE 1. Examples of super fiber bundles of geometric objects in the category \mathfrak{Q}_S

Fiber bundle of geometric objects derived from a fiber bundle of geometric objects $(\pi : W \rightarrow M; \mathbb{B})$	
Name	Definition
Tangent bundle	$(\pi : TM \rightarrow M, \mathbb{B} \equiv T)$
Cotangent bundle	$(\pi : T^*M \rightarrow M, \mathbb{B} \equiv T^*)$
Bundle of tensors	$(\pi : T_s^r M \rightarrow M, \mathbb{B} \equiv T_s^r)$
Full quantum tangent bundle	$(\pi : \hat{T}M \rightarrow M, \mathbb{B} \equiv \hat{T})$
Full quantum cotangent bundle	$(\pi : T^+M \rightarrow M, \mathbb{B} \equiv T^+)$
Dot bundle of tensors	$(\pi : \dot{T}_s^r M \rightarrow M, \mathbb{B} \equiv \dot{T}_s^r)$
Bundle of derivative of sections	$(\pi_1 : JD(W) \rightarrow M; \mathbb{B}^{(1)})$ $\mathbb{B}^{(1)} \equiv JD(-) \equiv Hom_{\mathbb{R}}(T(-); T\mathbb{B}(-)) _{JD(W)}$
pull-back of $Ds, \pi_1 \circ Ds = id_M$	$\phi^* Ds = Hom_{\mathbb{R}}(T(\phi); T\mathbb{B}(\phi^{-1})) \circ Ds \circ \phi$

Super fiber bundle of geometric objects derived from a super bundle of geometric objects $(\pi_B : B \rightarrow M \leftarrow W : \pi_W; \mathbb{B})$	
Name	Definition
Quantum bundle of derivative of sections	$(\pi_B : B \rightarrow M \leftarrow J\hat{D}(W); \pi_1; \mathbb{B}^{(1)})$ $\mathbb{B}^{(1)} \equiv J\hat{D}(-) \equiv Hom_Z(T(-); T\mathbb{B}(-)) _{J\hat{D}(W)}$
pull-back of $Ds, \pi_1 \circ Ds = id_M$	$\phi^* Ds = Hom_Z(T(\phi); T\mathbb{B}(\phi^{-1})) \circ Ds \circ \phi$

- i) if $B|_U \in Ob(\mathcal{C}(B)) \Rightarrow \mathbb{B}(B|_U) = \pi^{-1}_W(U) \in Ob(\mathcal{C}(W))$;
ii) if $f \in Hom(\mathcal{C}(B))$ with $f \equiv (f_B, f_M) : B|_U \rightarrow B|_{U'}$, then $\mathbb{B}(f) \in Hom(\mathcal{C}(W))$ and satisfies:
iii) $\pi_W \circ \mathbb{B}(f) = f_M \circ \pi_W$;
iv) if $B|_U \in Ob(\mathcal{C}(B))$, $\bar{U} \subset U \Rightarrow \mathbb{B}(f)|_{\pi_W^{-1}(\bar{U})} = \mathbb{B}(f|_{B|_{\bar{U}}})$.
 W is called the total bundle and B the base bundle. A section of π_W is called a quantum superfield of geometric objects.
The situation is resumed in the commutative diagram in (1).

$$(1) \quad \begin{array}{ccccc} B|_U & \xrightarrow{f_B} & B|_{\bar{U}} & \xrightarrow{\pi_B} & W|_U & \xrightarrow{\mathbb{B}(f)} & W|_{\bar{U}} \\ & \searrow \pi_B & & \swarrow \pi_W & & \searrow \pi_W & \\ & & U & \xrightarrow{f_M} & \bar{U} & & \end{array}$$

Example 2.3 (G -structures in the category \mathfrak{Q}_S). G -structures in the category \mathfrak{Q}_S are reductions of principal bundles of $(r|s)$ -frames, on a $(m|n)$ -dimensional quantum supermanifold, natural generalizations of analogous structures on commutative manifolds [59].

Example 2.4. In Tab. 1 are reported some further distinguished examples of (super)-bundles of geometric objects in the category \mathfrak{Q}_S .

In the physical applications it is important to consider the following.

Theorem 2.5. Let $(P, M, \pi; G)$ be a principal fiber bundle with structure group G in the category \mathfrak{Q}_S . Let $W \equiv P \times F/G$ be a fiber bundle associated to P with fibre F . Then, there exists a canonical covariant functor \mathbb{B} such that $(P, W; \mathbb{B})$ is a superbundle of geometric objects if we restrict the category $\mathcal{C}(P)$, where \mathbb{B} is defined, to the subcategory $\mathcal{C}(P)_\bullet$, where $Hom_{\mathcal{C}(P)_\bullet}(P|_U, P|_{U'})$ is the set of fibered quantum

diffeomorphisms $P|U \rightarrow P|U'$, such that the isomorphism on the structure group is the identity.

Proof. The proof can be copied by the classical case [61].⁵ \square

Definition 2.6 (Quantum Lie derivative of section of super bundle of geometric objects). *Let $(\phi_{M,\lambda}, \phi_{B,\lambda})_{\lambda \in \mathbb{R}}$, be a (local) 1-parameter group of (local) fiber bundle diffeomorphisms in the category \mathfrak{Q}_S . Let (ξ, ζ) the corresponding couple of quantum vector fields such that the fiber bundle (2) is commutative.*

$$(2) \quad \begin{array}{ccc} B|_U & \xrightarrow{\zeta} & TB|_U \\ \pi_B \downarrow & & \downarrow T(\pi_B) \\ U & \xrightarrow{\xi} & TU \end{array}$$

We call quantum Lie derivative of a section s of π_W , with respect to $(\phi_{M,\lambda}, \phi_{B,\lambda})_{\lambda \in \mathbb{R}}$, or (ξ, ζ) , the infinitesimal variation, $\partial \tilde{s}$, of the pull-back $\tilde{s}_\lambda \equiv \phi_\lambda^* s = \mathbb{B}(\phi_\lambda^{-1}) \circ s \circ \phi_{M,\lambda}$, with $\tilde{s} : \mathbb{R} \times M \rightarrow W$. (Here, for sake of simplicity we have denoted \tilde{s} globally define, but in general it is only locally defined.) Therefore we get

$$(3) \quad \mathcal{L}_\zeta s \equiv \partial \tilde{s} = \frac{d}{d\lambda}(\tilde{s}_\lambda)|_{\lambda=0} : M \rightarrow s^*vTW.$$

Definition 2.7 (Full quantum Lie derivative of section of super bundle of geometric objects). *Let $(\phi_{M,\lambda}, \phi_{B,\lambda})_{\lambda \in A}$, be a (local) 1-parameter group of (local) fiber bundle diffeomorphisms in the category \mathfrak{Q}_S . Let (ξ, ζ) the corresponding couple of vector fields such that the fiber bundle (4) is commutative.*

$$(4) \quad \begin{array}{ccc} B|_U & \xrightarrow{\zeta} & \hat{T}B|_U \equiv \text{Hom}_Z(A; TB|_U) \\ \pi_B \downarrow & & \downarrow \hat{T}(\pi_B) \equiv \text{Hom}_Z(1_A; T(\pi_B)) \\ U & \xrightarrow{\xi} & \hat{T}U \equiv \text{Hom}_Z(A; TU) \end{array}$$

We call full quantum Lie derivative of a section s of π_W , with respect to $(\phi_{M,\lambda}, \phi_{B,\lambda})_{\lambda \in A}$, or (ξ, ζ) , the infinitesimal variation, $\partial \tilde{s}$, of the pull-back $\tilde{s}_\lambda \equiv \phi_\lambda^* s = \mathbb{B}(\phi_\lambda^{-1}) \circ s \circ \phi_{M,\lambda}$, with $\tilde{s} : A \times M \rightarrow W$. (Here, for sake of simplicity we have denoted \tilde{s} globally define, but in general it is only locally defined.) Therefore we get

$$(5) \quad \mathcal{L}_\zeta s \equiv \partial \tilde{s} = \frac{d}{d\lambda}(\tilde{s}_\lambda)|_{\lambda=0} : M \rightarrow s^*v\hat{T}W \equiv \text{Hom}_Z(A; s^*vTW).$$

Proposition 2.8. *Under the same hypotheses of Definition 5 (resp. Definition 2.7) we get that whether $\pi_W : W \rightarrow M$ is a vector bundle, then we get $\mathcal{L}_\zeta s \equiv \partial \tilde{s} : M \rightarrow W$, (resp. $\mathcal{L}_\zeta s \equiv \partial \tilde{s} : M \rightarrow \hat{W} \equiv \text{Hom}_Z(A; W)$).*

Theorem 2.9. *Let $\hat{E}_k \subset J\hat{D}^k(W)$, be a quantum super PDE, with $(\pi_B : B \rightarrow M \leftarrow W : \pi_W; \mathbb{B})$ a super bundle of geometric objects. Let us assume that the map $\pi_{k,0} : \hat{E}_k \rightarrow W$ be surjective, with respect to the natural mapping $\pi_{k,0} : J\hat{D}^k(W) \rightarrow$*

⁵Warn ! Do not consider the concept of super bundle of geometric objects as synonym of fiber bundle in the category \mathfrak{C}_S of supermanifolds. (For a geometric theory of PDEs in \mathfrak{C}_S , see [60].)

W , and let assume also that \hat{E}_k is formally integrable and completely integrable. Then for any r -prolongation $(\hat{E}_k)_{+r}$, $r \geq 0$, we get the structure of super bundle of geometric objects reported in (6).

$$(6) \quad (\pi_B : B \rightarrow M \leftarrow (\hat{E}_k)_{+r} : \pi_{k+r}; \mathbb{B}^{(k+r)} = J\hat{D}^{k+r}(-)_\bullet).$$

where $J\hat{D}^{k+r}(-)_\bullet \equiv J\hat{D}^{k+r}(\mathbb{B}_\bullet(-))|_{\hat{E}_{k+r}}$, with $\mathbb{B}_\bullet(-)$ the restriction of $\mathbb{C}(B)(-)$ to the sub-category $\mathcal{C}_\bullet(B) \subset \mathcal{C}(B)$, such that the corresponding morphisms $f \in \text{Hom}(\mathcal{C}_\bullet(B))$ are such that $J\hat{D}^{k+r}(\mathbb{B}_\bullet(f))_\bullet$ are symmetries of \hat{E}_{k+r} . We call (6) a $(k+r)$ -holonomic super bundle of geometric objects.

Proof. This means that for any section $u : M \rightarrow (\hat{E}_k)_{+r}$, we can calculate its quantum Lie derivative, with respect to a 1-group of (local) fiber bundle diffeomorphisms of $\pi_B : B \rightarrow M$, that should be also symmetries of \hat{E}_k , hence of $(\hat{E}_k)_{+r}$ too. In particular for holonomic sections $u = D^{k+r}s$, namely s is a solution of \hat{E}_k , we get $\partial D^{k+r}s : M \rightarrow (D^{k+r}s)^*vTJ\hat{D}^{k+r}(W) \cong J\hat{D}^{k+r}(s^*vTW)$, such that the diagram (7) is commutative.

$$(7) \quad \begin{array}{ccc} (D^{k+r}s)^*vTJ\hat{D}^{k+r}(W) & \xlongequal{\sim} & J\hat{D}^{k+r}(s^*vTW) \\ \uparrow & & \parallel \\ \hat{E}_{k+r}[s] \equiv (D^{k+r}s)^*vT(\hat{E}_k)_{+r} & \xhookrightarrow{\quad} & J\hat{D}^{k+r}(E[s]) \end{array}$$

Therefore the quantum Lie derivative of a solution of \hat{E}_k can be identified with a solution of the linearized equation $\hat{E}_k[s] \subset J\hat{D}^k(E[s])$, with $E[s] \equiv s^*vTW$.⁶

□

Theorem 2.10 (Classification of quantum supermanifolds with classic limit). *We can classify quantum supermanifolds, having a classic limit, $\pi_C : M \rightarrow M_C$, on the ground of its classic limit M_C . In other words, we can say M has some property whether its classic limit has this property. (For example we can say that M is orientable when M_C is so.) In Tab. 2 are reported some quantum supermanifolds classified with respect to the holonomic group of their classic limit.*

Proof. This follows from standard properties of classifications of fiber bundles in algebraic topology, (see e.g., [94]), and taking into account that each fiber over $p \in M_C$ is contractible to a point. □

Example 2.11. *Quantum (super)manifolds with classic limits having nontrivial structure can be considered of particular interest whether from the mathematical point of view or physical one. In particular, recent works investigate on Calabi-Yau mirror-pairs of submanifolds of G_2 -manifolds and $\text{Spin}(7)$ -manifolds. (See e.g., [4].)⁷ These are Riemannian manifolds classified on the ground of their holonomy groups. Let us recall in Tab. 1 the classification of possible holonomy*

⁶Warn ! The set of all solutions of the linearized equation $\hat{E}_k[s] \equiv (D^k s)^*vT\hat{E}_k$ is larger than ones obtained as Lie derivative of s , with respect to some 1-parameter symmetry group of some super bundle of geometric objects $(\pi_B : B \rightarrow M \leftarrow W : \pi_W; \mathbb{B})$.

⁷Calabi-Yau manifolds are compact, complex Kähler manifolds that have trivial first Chern classes (over \mathbb{R}). Yau proved (1977-1979) a conjecture by Calabi (1957) that there exists on every CY-manifold a Kähler metric with vanishing Ricci curvature [102, 13]. One important class of G_2 manifold are the ones from CY-manifolds. Let (X, ω, Ω) be a complex 3-dimensional CY-manifold with Kähler form ω and a nowhere vanishing holomorphic 3-form Ω , then $X^6 \times S^1$ has holonomy

groups $\text{Hol}(M_C, g)$ for simply connected Riemannian manifolds (M_C, g) which are not locally a product space and not locally a Riemannian symmetric space (shortly denote them by INS-SCRM here). (This classification was first obtained by M. Berger (1955) and further improved by some other mathematicians (D. Alekseevski, Brown-Gray, Hitchin ...). R.L. Bryant (1987) first proved existence of metrics with holonomy G_2 (resp. $\text{Spin}(7)$) on 7-dimensional manifolds (resp. 8-dimensional manifolds) [6, 11, 24, 33].

The group G_2 can be identified with the group of automorphisms of octonions \mathbb{O} or equivalently the subgroup of $GL(7, \mathbb{R})$ that preserves a suitable 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$, given in (8).

$$(8) \quad \begin{cases} \varphi_0 &= \frac{1}{6} \epsilon_{ijk} dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7) \\ &= dx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge dx^4 \wedge dx^5 + dx^1 \wedge dx^6 \wedge dx^7 + dx^2 \wedge dx^4 \wedge dx^6 \\ &\quad - dx^2 \wedge dx^5 \wedge dx^7 - dx^3 \wedge dx^4 \wedge dx^7 - dx^3 \wedge dx^5 \wedge dx^6. \end{cases}$$

A smooth 7-dimensional manifold M_C has a G_2 -structure if its tangent bundle reduces to a G_2 bundle, or equivalently if there is a 3-form $\varphi \in \Omega^3(M_C)$ such that at each $p \in M_C$ the pair $(T_p M_C, \varphi(p))$ is isomorphic to $(T_0 \mathbb{R}^7, \varphi_0)$. Then the 3-form φ identifies an orientation $\mu \in \Omega^7(M_C)$, and μ determines a metric g_φ . A manifold with G_2 -structure (M_C, φ) , is called a G_2 manifold if $\text{Hol}(M_C, g_\varphi) \subset G_2$, or equivalently $\nabla_{g_\varphi} \varphi = 0 \Leftrightarrow \{d\varphi = 0, d(\star_{g_\varphi} \varphi) = 0\}$. (Because G_2 is a connected, simply connected group, a connected 7-dimensional manifold with a G_2 -structure is orientable and admits a spin structure, i.e., its first two Stiefel-Whitney classes vanish. (Gray (1969) and R. Bryant (2005).) G_2 manifolds can be also characterized as critical points of a suitable functional on the 3-forms (N. Hitchin (2000).).

In [4] it is defined mirror pair a couple of Calabi-Yau manifolds if their complex structures are induced from the same calibration 3-form, φ , in a G_2 manifold. It assigns to a G_2 manifold (M_C, φ, Λ) , with the calibration 3-form φ and a oriented 2-plane field Λ , a pair of parametrized tangent bundle valued 2- and 3-forms of M_C . These forms can be used to define different complex and symplectic structures on certain 6-dimensional subbundles of TM_C . More precisely, let (M_C, φ) be a G_2 manifold. A 4-dimensional submanifold $X \subset M_C$ is called coassociative if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M_C$ is called associative if $\varphi|_Y = \text{vol}(Y)$. This condition is equivalent to the condition $\chi|_Y = 0$, where $\chi \in \Omega^3(M, TM) \equiv C^\infty(TM \otimes \Lambda_3^0 M_C)$ is the TM_C -valued 3-form on M_C given by $\langle \chi(u, v, w), z \rangle = \star \varphi(u, v, w, z)$. We can also define a TM -valued 2-form $\psi \in C^\infty(TM_C \otimes \Lambda_2^0 M_C)$, given by $\langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle$. To any 3-dimensional submanifold $Y \subset (M_C, \varphi)$, χ assigns a normal vector field, which vanishes when Y is associative. For any associative manifold $Y \subset (M_C, \varphi)$ with a non-vanishing oriented 2-plane field, χ defines a complex structure on its normal bundle. In particular, any coassociative submanifold $X \subset M_C$ has an almost complex structure if its normal bundle has a non vanishing section. Two CY-manifolds are mirror pairs of each other if their complex structures are induced from the same calibration 3-form in a G_2 manifold. Furthermore, we call them strong mirror pairs if their normal vector fields ξ and ξ' are homotopic to each other through non-vanishing vector fields. One can assign to a G_2 manifold (M_C, φ, Λ) , with the

group $SU(3) \subset G_2$, hence is a G_2 manifold. In this case $\varphi = \Re \Omega + \omega \wedge dt$. Similarly, $X^6 \times \mathbb{R}$ gives a noncompact G_2 manifold.

calibration 3-form φ and oriented 2-plane field Λ , a pair of parametrized tangent bundle valued 2- and 5-forms of M_C . These forms can be used to define differential complex and symplectic structures on certain 6-dimensional subbundles of TM_C . When these bundles are integrated they give mirror CY-manifolds. In a similar way one can recognize mirror dual G_2 manifolds inside of a $Spin(7)$ manifold (M_C^8, ψ) . In case M_C admits an oriented 3-plane field, by iterating this process one can obtain Calabi-Yau submanifolds pairs in M_C whose complex and symplectic structures determine each other via the calibration form of the ambient G_2 (or $Spin(7)$) manifold.

TABLE 2. Classification of quantum supermanifolds on the ground of the Holonomy groups of their classic limit $\pi_C : M \rightarrow M_C$.

$\dim_{\mathbb{R}} M_C$	$Hol(M_C, g)$	Manifold's name	Further characterizations
n	$SO(n)$	Orientable	
$2n$	$U(n)$	Kähler	orientable: $[U(n) \subset SO(2n)]$
$2n$	$SU(n)$	Calabi-Yau	Ricci-flat, Kähler, orientable: $[SU(n) \subset U(n) \subset SO(2n)]$
$4n$	$Sp(n).Sp(1)$	Quaternionic-Kähler	Einstein
$4n$	$Sp(n)$	Hyperkähler	Ricci-flat, Calabi-Yau, Kähler, orientable: $[Sp(n) \subset SU(2n) \subset U(2n) \subset SO(4n)]$
7	G_2	G_2 -manifold	Ricci-flat, spin, orientable: $[G_2 \subset SO(7)]$
8	$Spin(7)$	$Spin(7)$ -manifold	Ricci-flat

A manifold with G_2 structure (M_C, φ) is called a G_2 manifold if $Hol(M_C, g_\varphi) \subset G_2$, or equivalently $\nabla^{(g_\varphi)} \varphi = 0$.

Theorem 2.12 (Quantum super manifolds with structured classic limit as super bundles of geometric objects). *Let M be a quantum super manifolds with structured classic limits, $\pi_C : M \rightarrow M_C$. A super bundle of geometric objects on M , $(\pi_C : B \rightarrow M \leftarrow W : \pi_W)$ identifies a super bundle of geometric objects on M_C iff $Hom(\mathcal{C}(B))$ and $Hom(\mathcal{C}(W))$ are restricted to (local) fiber bundles morphisms and the functor \mathbb{B} admits such a restriction, such that the diagrams in (9) are commutative.*

$$(9) \quad \begin{array}{ccc} B & \xrightarrow{f_B} & B \\ \pi_B \downarrow & & \downarrow \pi_B \\ M & \xrightarrow{f_B} & M \\ \pi_C \downarrow & & \downarrow \pi_C \\ M_C & \xrightarrow{f_C} & M_C \end{array} \quad \begin{array}{ccc} W & \xrightarrow{f_W} & W \\ \pi_W \downarrow & & \downarrow \pi_W \\ M & \xrightarrow{f_B} & M \\ \pi_C \downarrow & & \downarrow \pi_C \\ M_C & \xrightarrow{f_C} & M_C \end{array}$$

Then, when above conditions are satisfied, we say that the super bundle of geometric objects on M , $(\pi_C : B \rightarrow M \leftarrow W : \pi_W)$, is a classic-regular super bundle of geometric objects on the quantum supermanifold M .

Proof. In fact any super bundle of geometric object on M , i.e., $(\pi_C : B \rightarrow M \leftarrow W : \pi_W; \mathbb{B})$, one has the natural fiber bundle structures on M_C reported in (10).

$$(10) \quad \begin{array}{ccc} B & & W \\ \pi_B \swarrow & & \nwarrow \pi_W \\ & M & \\ \pi_C \downarrow & & \downarrow \pi_C \\ M_C & & M_C \end{array} \quad \begin{array}{c} \pi_{B,C} \swarrow \\ \searrow \pi_{W,C} \end{array}$$

Therefore for any morphism $(f) = (f_B, f_M, f_{M_C})$ must be the diagram in (11) commutative.

$$(11) \quad \begin{array}{ccccc} B & \xrightarrow{f_B} & B & & W \\ \pi_B \searrow & & \swarrow \pi_B & & \nwarrow \pi_W \\ & M & & M & \\ \pi_C \downarrow & & \swarrow \pi_{B,C} & & \nwarrow \pi_{W,C} \\ M_C & \xrightarrow{f_C} & M_C & & M_C \end{array} \quad \begin{array}{c} \pi_W \swarrow \\ \nwarrow \pi_W \\ \pi_C \downarrow \\ \pi_C \downarrow \end{array} \quad \begin{array}{c} \mathbb{B}(f) \\ f_M \\ f_C \end{array}$$

□

Example 2.13. The $(k+r)$ -holonomic super bundle of geometric objects $(\pi_B : B \rightarrow M \leftarrow (\hat{E}_k)_{+r} : \pi_{k+r}; \mathbb{B}^{(k+r)} = J\hat{D}^{k+r}(-))$ is a classic-regular super bundle of geometric objects on the quantum supermanifold M .

Remark 2.14 (χ -flow). In [4] it is considered the spinor bundle $Q^7 \rightarrow S^3$ (with \mathbb{C}^2 fibers). Q is just a G_2 manifold where the two 6-dimensional submanifolds $S^2 \times \mathbb{R}^4 \subset Q$ and $S^3 \times \mathbb{R}^3 \subset Q$ constitute a mirror pair in the sense introduced in this paper. The zero section $S^3 \subset Q$ is an associative submanifold, i.e., $\varphi|_{S^3} \equiv \text{vol}(S^3)$. This condition is equivalent to the condition $\chi|_{S^3} \equiv 0$, where $\chi \in C^\infty(TQ \otimes \Lambda_3^0(Q))$, is defined by $\langle \chi(u, v, w), z \rangle = (\star_{g_\varphi} \varphi)(u, v, w, z)$. The equivalence of these conditions follows from $\varphi(u, v, w)^2 + |\chi(u, v, w)|^2/4 = |u \wedge v \wedge w|^2$. Then χ identifies a flow on a 3-dimensional submanifold $f : Y \rightarrow (Q, \varphi)$, called χ -flow, described by $(\partial_t f) = \chi(f_* \text{vol}(Y))$. (See also some previous works by R. L. Bryant & M-S. Salamon (1989) and N. Hitchin (2000).) Furthermore, in [4] the following conjecture is made.

Conjecture. Since one can imbed any homotopy 3-sphere Σ^3 into Q (homotopic to the zero-section), one can conjecture that the χ -flow on $\Sigma^3 \subset Q$, takes Σ^3 diffeomorphically onto the zero section S^3 .

This conjecture is justified since $\Sigma^3 \cong S^3$, as it is nowadays well proved. (See [79, 80, 82].) In fact, the PDE's algebraic topology as introduced by A. Prástaro, to characterize global solutions of PDE's, can be used also to solve this conjecture when applied to the χ -flow equation.

More explicitly we have the following lemma.

Lemma 2.15. The chi-flow equation admits a solution that diffeomorphically relates any 3-dimensional homotopy sphere Σ^3 , with S^3 .

Proof. Let us consider the following fiber bundle $\pi : W \equiv \mathbb{R} \times Q^7 \rightarrow \mathbb{R} \times S^3$, $(t, x^k)_{1 \leq k \leq 7} \mapsto (t, x^k)_{1 \leq k \leq 3}$. Here the fiber is $\mathbb{C}^2 \cong \mathbb{R}^4$. One has the canonical embedding of S^3 into W , for any $t \in \mathbb{R}$: $(t, x^k)_{1 \leq k \leq 3} \rightarrow (t, x^1, x^2, x^3, 0, \dots, 0)$, identified by the zero section of π . In (12) are given the local representations of the geometric objects above introduced.

$$(12) \quad \left\{ \begin{array}{lcl} \varphi & = & \frac{1}{6}\epsilon_{ijk}dx^i \wedge dx^j \wedge dx^k \\ & = & dx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge dx^4 \wedge dx^5 + dx^1 \wedge dx^6 \wedge dx^7 \\ & + & dx^2 \wedge dx^4 \wedge dx^6 - dx^2 \wedge dx^5 \wedge dx^7 - dx^3 \wedge dx^4 \wedge dx^7 - dx^3 \wedge dx^5 \wedge dx^6 \\ g_\varphi & = & \delta_{ij}dx^i \otimes dx^j \\ \mu & = & dx^1 \wedge \dots \wedge dx^7 \\ \star_\varphi \varphi & = & \frac{1}{24}\epsilon_{ijkl}dx^i \wedge dx^j \wedge dx^k \wedge dx^l \\ & = & dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7 + dx^2 \wedge dx^3 \wedge dx^6 \wedge dx^7 + dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \\ & + & dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^7 - dx^1 \wedge dx^3 \wedge dx^4 \wedge dx^6 - dx^1 \wedge dx^2 \wedge dx^5 \wedge dx^6 \\ & - & dx^1 \wedge dx^2 \wedge dx^4 \wedge dx^7 \\ \partial x_i \wedge \partial x_j & = & \epsilon_{ijk} \partial x_k, \text{ (cross product).} \\ \chi & = & \chi_{ijk}^p \partial x_p \otimes dx^i \wedge dx^j \wedge dx^k \\ & = & \frac{1}{24}\epsilon_{ijk} p \partial x_p \otimes dx^i \wedge dx^j \wedge dx^k \end{array} \right.$$

In (13) is given the χ -flow equation in local coordinates.

$$(13) \quad \left\{ \begin{array}{l} E_1 \subset JD(W) \quad \left\{ (\partial t.f^p) = \sum_{1 \leq p \leq 7} \frac{1}{24} \det(\partial x_i.f^j) \epsilon_{123}^p \right\}_{1 \leq i,j \leq 3} : \\ (\partial t.f^1) = 0 \\ (\partial t.f^2) = 0 \\ (\partial t.f^3) = 0 \\ (\partial t.f^s) = \frac{1}{24}(\det(\partial x_i.f^j))_{1 \leq i,j \leq 3} \quad 4 \leq s \leq 7. \end{array} \right.$$

Since Σ^3 is diffeomorphic to S^3 , let us consider $\phi : \Sigma^3 \rightarrow S^3$ this diffeomorphism. Then a solution of equation (13) is written in (14).

$$(14) \quad \left\{ \begin{array}{l} f^k(t, x^1, \dots, x^7) = \phi^k(x^1, x^2, x^3) \quad 1 \leq k \leq 3. \\ f^s(t, x^1, \dots, x^7) = \frac{1}{24}j(\phi)t \end{array} \right.$$

where $j(\phi)$ is the determinant of the jacobian of diffeomorphism ϕ . Therefore, the proof of the lemma is down. \square

Theorem 2.16 (Quantum χ -flow). *We define quantum \hat{Q}^7 any quantum (super)manifold M having as classic limit $M_C = Q^7$. We say quantum χ -flow on M , any quantum flow that projects on the classic limit in a chi-flow. Then we have that a quantum χ -flow represents diffeomorphically any homotopy 3-sphere, $\Sigma^3 \subset Q^7$, onto $S^3 \subset Q^7$.*

Proof. This follows directly from above definitions and Lemma 2.15. \square

Theorem 2.17. *A gauge theory in the category \mathfrak{Q}_S is fully covariant.*

Proof. In fact, any gauge theory in the category \mathfrak{Q}_S , can be identified by the following super bundle of geometric objects on a quantum supermanifold M :

$$(\pi_P : P \rightarrow M \leftarrow Hom_Z(TM; \mathfrak{g}) : \pi; \mathbb{B}(-) = Hom_Z(-; 1_{\mathfrak{g}})),$$

where $\pi_P : P \rightarrow M$ is a G -principal bundle bundle over M , in the category \mathfrak{Q}_S , and \mathfrak{g} is the Lie superalgebra corresponding to the quantum Lie supergroup G of the P . Therefore one has the commutative diagram reported in (15).

$$(15) \quad \begin{array}{ccccc} P \equiv B & & & & W \equiv \text{Hom}_Z(TM; \mathfrak{g}) \\ & \searrow \pi_B & & \swarrow \pi_W & \\ & M & & & \\ & \downarrow \pi_C & & & \\ & M_C & & & \end{array}$$

$\pi_{B,C}$ (curved arrow from B to M_C) $\pi_{W,C}$ (curved arrow from W to M_C)

This is a classic-regular super bundle of geometric objects on M . (For details on gauge theory in the category \mathfrak{Q}_S see Refs. [70, 75, 77, 78].) \square

In the following we generalize the definition of instanton and soliton of the classical field theory.

Definition 2.18 (Quantum super-instantons and quantum super-solitons). *A quantum super-instanton is a solution of the quantum super Yang-Mills equation (\widehat{YM}) with non-trivial topology.*

A quantum super-soliton is a sectional compact quantum super-instanton.

Remark 2.19. *Let us emphasize that the classic limits of quantum supermanifolds considered in Definition 2.18 do not necessarily coincide with usual instantons and solitons respectively. (Compare with the definitions usually adopted in commutative differential geometry.⁸ Compare also with their non-commutative extension given in [8, 50].)*

3. Conservation Laws in Quantum Super PDEs

Conservation laws are considered for PDE's built in the category \mathfrak{Q}_S of quantum supermanifolds. These are functions defined on the integral bordism groups of such equations and belonging to suitable Hopf algebras (*full quantum Hopf algebras*). In particular, we specialize our calculations on the quantum super Yang-Mills equations and quantum black holes.

In this section we shall resume some our fundamental definition and result for PDE's in the *category of quantum supermanifolds*, \mathfrak{Q}_S , where the objects are just quantum supermanifolds, and the morphisms are maps of class Q_w^k , $k \in \{0, 1, 2, \dots, \infty, \omega\}$ [67, 68, 70, 73, 76, 77]. A small subcategory is $\mathfrak{C}_S \subset \mathfrak{Q}_S$ of supermanifolds as defined in [60].

Let $\pi : W \rightarrow M$ be a fiber bundle, in the category \mathfrak{Q}_S , such that $\dim W = (m|n, r|s)$, over the quantum superalgebra $B \equiv A \times E$ and $\dim M = (m|n)$ over A and such that E is a Z -module, with $Z \equiv Z(A) \subset A$, the center of A . The *quantum k -jet-derivative space* $J\hat{D}^k(W)$ of $\pi : W \rightarrow M$, is the k -jet-derivative space of sections of π , belonging to the class Q_w^k . The k -jet-derivative $J\hat{D}^k(W)$

⁸These represent solutions of the Yang-Mills equation on four-dimensional Euclidean space, considered as the Wick rotation of Minkowski spacetime. (Let us recall the pioneering result by A. M. Polyakov [53] proving that instantons effects in 3-dimensional QED, coupled to a scalar field (i.e., Higgs field) lead to a massive photon.) In a paper, with the title "*Super bundle of geometric objects, Yang-Mills gauge field and Spin^G-instantons*", announced in [87], and appeared in Section 4.8 of the book [61], were developed some relations between super bundle of geometric objects on classical instantons and gravitational instantons. In particular were there applied the fully covariant theory of classical spinor fields, first previously developed in [57, 58].

is a quantum supermanifold modeled on the quantum superalgebra, (*quantum k-holonomic superalgebra*), reported in (16).

$$(16) \quad \begin{cases} B_k \equiv \prod_{0 \leq s \leq k} \left(\prod_{i_1 + \dots + i_s \in \mathbb{Z}_2, i_r \in \mathbb{Z}_2} \hat{A}_{i_1 \dots i_s}^s(E) \right) \\ \hat{A}_{i_1 \dots i_s}^s(E) \equiv \text{Hom}_Z(A_{i_1} \otimes_Z \dots \otimes_Z A_{i_s}; E) \\ \hat{A}_0(E) \equiv A \times E \\ \hat{A}_i(E) \equiv \hat{A}_0(E) \times \hat{A}_1(E) \equiv \text{Hom}_Z(A_0; E) \times \text{Hom}_Z(A_1; E). \end{cases}$$

Each $\hat{A}_{i_1 \dots i_s}^s(E)$ is a quantum superalgebra with \mathbb{Z}_2 -gradation induced by E . Hence $\hat{A}_{i_1 \dots i_s}^s(E)_q \equiv \text{Hom}_Z(A_{i_1} \otimes_Z \dots \otimes_Z A_{i_s}; E_p)$, $i_r, p, q \in \mathbb{Z}_2$, $q \equiv i_1 + \dots + i_s + p$. If $(x^A, y^B)_{1 \leq A \leq m+n, 1 \leq B \leq r+s}$ are fibered quantum coordinates on the quantum supermanifold W over M , then $(x^A, y^B, y_A^B, \dots, y_{A_1 \dots A_k}^B)$ are fibered quantum coordinates on $J\hat{D}^k(W)$ over M , with the following gradations: $|x^A| = |A|$, $|y^B| = |B|$, $|y_{A_1 \dots A_s}^B| = |B| + |A_1| + \dots + |A_s|$. Note, also, that there is not symmetry in the indexes A_i . $J\hat{D}^k(W)$ is an affine bundle over $J\hat{D}^{k-1}(W)$ with associated vector bundle $\pi_{k,0}^* \text{Hom}_Z(\hat{S}_0^k M; vTW)$, where $\hat{S}_0^k M$ is the k -times symmetric tensor product of TM , considered as a bundle of Z -modules over M , and $\pi_{k,0} : J\hat{D}^k(W) \rightarrow W$ is the canonical surjection. Another important example is $\hat{J}_{m|n}^k(W)$, that is the k -jet space for quantum supermanifolds of dimension $(m|n)$ (over A) contained in the quantum supermanifold W . This quantum supermanifold locally looks like $J\hat{D}^k(W)$. Set $J\hat{D}^\infty(W) \equiv \lim_{\leftarrow k} J\hat{D}^k(W)$, $\hat{J}_{m|n}^\infty(W) \equiv \lim_{\leftarrow k} \hat{J}_{m|n}^k(W)$. These are quantum supermanifolds modeled on $B \equiv \prod_k B_k$.

A *quantum super PDE* of order k on the fibre bundle $\pi : W \rightarrow M$, defined in the category of quantum supermanifolds, \mathfrak{Q}_S , is a subset $\hat{E}_k \subset J\hat{D}^k(W)$, or $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$. A geometric theory of quantum (super) PDE's can be formulated introducing suitable hypotheses of regularity on \hat{E}_k . (See [70].)

The characterization of global solutions of a PDE $\hat{E}_k \subseteq \hat{J}_{m|n}^k(W)$, in the category \mathfrak{Q}_S , can be made by means of its integral bordism groups $\Omega_{p|q}^{\hat{E}_k}$, $p \in \{0, 1, \dots, m-1\}$, $q \in \{0, 1, \dots, n-1\}$. Let us shortly recall some fundamental definitions and results about. Let $f_i : X_i \rightarrow \hat{E}_k$, $f_i(X_i) \equiv N_i \subset \hat{E}_k$, $i = 1, 2$, be $(p|q)$ -dimensional admissible compact closed smooth integral quantum supermanifolds of \hat{E}_k . The admissibility requires that N_i should be contained into some solution $V \subset \hat{E}_k$, identified with a (m, n) -chain, with coefficients in A . Then, we say that they are \hat{E}_k -bordant if there exists a $(p+1|q+1)$ -dimensional smooth quantum supermanifolds $f : Y \rightarrow \hat{E}_k$, such that $\partial Y = X_1 \sqcup X_2$, $f|_{X_i} = f_i$, $i = 1, 2$, and $V \equiv f(Y) \subset \hat{E}_k$ is an admissible integral quantum supermanifold of \hat{E}_k of dimension $(p+1|q+1)$. We say that N_i , $i = 1, 2$, are \hat{E}_k -quantum-bordant if there exists a $(p+1|q+1)$ -dimensional smooth quantum supermanifolds $f : Y \rightarrow \hat{J}_{m|n}^k(W)$, such that $\partial Y = X_1 \sqcup X_2$, $f|_{X_i} = f_i$, $i = 1, 2$, and $V \equiv f(Y) \subset \hat{J}_{m|n}^k(W)$ is an admissible integral manifold of $\hat{J}_{m|n}^k(W)$ of dimension $(p+1|q+1)$. Let us denote the corresponding bordism groups by $\Omega_{p|q}^{\hat{E}_k}$ and $\Omega_{p|q}(\hat{E}_k)$, $p \in \{0, 1, \dots, m-1\}$, $q \in \{0, 1, \dots, n-1\}$, called

respectively $(p|q)$ -dimensional integral bordism group of \hat{E}_k and $(p|q)$ -dimensional quantum bordism group of \hat{E}_k . Therefore these bordism groups work, for $(p, q) = (m-1, n-1)$, in the category of quantum supermanifolds that are solutions of \hat{E}_k . Let us emphasize that singular solutions of \hat{E}_k are, in general, (piecewise) smooth quantum supermanifolds into some prolongation $(\hat{E}_k)_{+s} \subset \hat{J}_{m|n}^{k+s}(W)$, where the set, $\Sigma(V)$, of *singular points* of a solution V is a non-where dense subset of V . Here we consider *Thom-Boardman singularities*, i.e., $q \in \Sigma(V)$, if $(\pi_{k,0})_*(T_q V) \not\cong T_q V$. However, in the case where \hat{E}_k is a differential equation of finite type, i.e., the symbols $\hat{g}_{k+s} = 0$, $s \geq 0$, then it is useful to include also in $\Sigma(V)$, discontinuity points, $q, q' \in V$, with $\pi_{k,0}(q) = \pi_{k,0}(q') = a \in W$, or with $\pi_k(q) = \pi_k(q') = p \in M$, where $\pi_k = \pi \circ \pi_{k,0} : \hat{J}_{m|n}^k(W) \rightarrow M$. We denote such a set by $\Sigma(V)_S$, and, in such cases we shall talk more precisely of *singular boundary* of V , like $(\partial V)_S = \partial V \setminus \Sigma(V)_S$. Such singular solutions are also called *weak solutions*. Let us define some notation to distinguish between some integral bordisms.

Definition 3.1. Let $\Omega_{m-1|n-1}^{\hat{E}_k}$, (resp. $\Omega_{m-1|n-1,s}^{\hat{E}_k}$, resp. $\Omega_{m-1|n-1,w}^{\hat{E}_k}$), be the integral bordism group for $(m-1|n-1)$ -dimensional smooth admissible regular integral quantum supermanifolds contained in \hat{E}_k , bordered by smooth regular integral quantum supermanifold-solutions, (resp. piecewise-smooth or singular solutions, resp. singular-weak solutions), of \hat{E}_k .

Theorem 3.2. [70] One has the exact commutative diagram (17). Therefore, one has the canonical isomorphisms:

$$\begin{aligned} K_{m-1|n-1,w/(s,w)}^{\hat{E}_k} &\cong K_{m-1|n-1,s}^{\hat{E}_k}; \quad \Omega_{m-1|n-1}^{\hat{E}_k}/K_{m-1|n-1,s}^{\hat{E}_k} \cong \Omega_{m-1|n-1,s}^{\hat{E}_k}; \\ \Omega_{m-1|n-1,s}^{\hat{E}_k}/K_{m-1|n-1,s,w}^{\hat{E}_k} &\cong \Omega_{m-1|n-1,w}^{\hat{E}_k}; \quad \Omega_{m-1|n-1}^{\hat{E}_k}/K_{m-1|n-1,w}^{\hat{E}_k} \cong \Omega_{m-1|n-1,w}^{\hat{E}_k}. \end{aligned}$$

If \hat{E}_k is formally quantum superintegrable, then one has the following isomorphisms:

$$\Omega_{m-1|n-1}^{\hat{E}_k} \cong \Omega_{m-1|n-1}^{\hat{E}_\infty} \cong \Omega_{m-1|n-1,s}^{\hat{E}_\infty}; \quad \Omega_{m-1|n-1,w}^{\hat{E}_k} \cong \Omega_{m-1|n-1,w}^{\hat{E}_\infty}.$$

$$(17) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_{m-1|n-1,w/(s,w)}^{\hat{E}_k} & \longrightarrow & K_{m-1|n-1,w}^{\hat{E}_k} & \longrightarrow & K_{m-1|n-1,s,w}^{\hat{E}_k} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_{m-1|n-1,s}^{\hat{E}_k} & \longrightarrow & \Omega_{m-1|n-1}^{\hat{E}_k} & \longrightarrow & \Omega_{m-1|n-1,s}^{\hat{E}_k} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \Omega_{m-1|n-1,w}^{\hat{E}_k} & \longrightarrow & \Omega_{m-1|n-1,w}^{\hat{E}_k} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Theorem 3.3. Let $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ be a quantum super PDE that is formally quantum superintegrable, and completely superintegrable. We shall assume that the symbols $\hat{g}_{k+s} \neq 0$, $s = 0, 1$. (This excludes the case $k = \infty$.) Then one has the following isomorphisms: $\Omega_{p|q,s}^{\hat{E}_k} \cong \Omega_{p|q,w}^{\hat{E}_k} \cong \Omega_{p|q}(\hat{E}_k)$, with $p \in \{0, \dots, m-1\}$ and $q \in \{0, \dots, n-1\}$.

Proof. In fact, in these cases any weak solution identifies a singular solution, by connecting its branches by means of suitable pieces of fibres. Furthermore, since \hat{E}_{k+1} is a strong retract of $\hat{J}_{m|n}^{k+1}(W)$, we can deform any quantum bording $V \subset \hat{J}_{m|n}^{k+1}(W)$, $\dim V = (m|n)$, with $\partial V \subset \hat{E}_{k+1}$, into a (singular) solution of \hat{E}_{k+1} , hence into a solution of \hat{E}_k . (For details see Refs. [70].) \square

Corollary 3.4. *Let $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ be a quantum super PDE, that is formally superintegrable and completely superintegrable. One has the following isomorphisms:*
 $\Omega_{m-1|n-1,w}^{\hat{E}_k} \cong \Omega_{m-1|n-1}(\hat{E}_k) \cong \Omega_{m-1|n-1,w}^{\hat{E}_{k+h}} \cong \Omega_{m-1|n-1,w}^{\hat{E}_\infty} \cong \Omega_{m-1|n-1,w}(\hat{E}_{k+h}) \cong \Omega_{m-1|n-1}(\hat{E}_\infty)$.

In order to distinguish between quantum integral supermanifolds V representing singular solutions, where $\Sigma(V)$ has no discontinuities, and quantum integral supermanifolds where $\Sigma(V)$ contains discontinuities, we can also consider "conservation laws" valued on quantum integral supermanifolds N representing the integral bordism classes $[N]_{\hat{E}_k} \in \Omega_{p|q}^{E_k}$.

Definition 3.5. *Let us define the space of quantum integral conservation laws of $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ the Z -module given in (18).*

$$(18) \quad \begin{cases} \hat{\mathcal{J}}(\hat{E}_k) & \equiv \oplus_{p,q \geq 0} \frac{\widehat{\Omega}^{p|q}(\hat{E}_k) \cap d^{-1}(C\widehat{\Omega}^{p+1|q+1}(\hat{E}_k))}{d\widehat{\Omega}^{p-1|q-1}(E_k) \oplus \{C\widehat{\Omega}^{p|q}(\hat{E}_k) \cap d^{-1}(C\widehat{\Omega}^{p+1|q+1}(\hat{E}_k))\}} \\ & \equiv \oplus_{p,q \geq 0} \hat{\mathcal{J}}(\hat{E}_k)^{p|q}. \end{cases}$$

Here $C\widehat{\Omega}^{p|q}(\hat{E}_k)$ denotes the space of all quantum $(p|q)$ -forms on \hat{E}_k . Then we define quantum integral characteristic supernumbers of N , with $[N]_{\hat{E}_k} \in \widehat{\Omega}_{p|q}^{\hat{E}_k}$, the numbers $\hat{i}[N] \equiv \langle [N]_{\hat{E}_k}, [\alpha] \rangle \in B$, for all $[\alpha] \in \hat{\mathcal{J}}(\hat{E}_k)^{p|q}$.

Then, one has the following theorems.

Theorem 3.6. [70] *Let us assume that $\hat{\mathcal{J}}(\hat{E}_k)^{p|q} \neq 0$. One has a natural homomorphism: $\underline{j}_{p|q} : \Omega_{p|q}^{\hat{E}_k} \rightarrow \text{Hom}_A(\hat{\mathcal{J}}(\hat{E}_k)^{p|q}; A)$, $[N]_{\hat{E}_k} \mapsto \underline{j}_{p|q}([N]_{\hat{E}_k})$, $\underline{j}_{p|q}([N]_{\hat{E}_k})([\alpha]) = \int_N \alpha \equiv \langle [N]_{\hat{E}_k}, [\alpha] \rangle$. Then, a necessary condition that $N' \in [N]_{\hat{E}_k}$ is the following: $\hat{i}[N] = \hat{i}[N']$, $\forall [\alpha] \in \hat{\mathcal{J}}(\hat{E}_k)^{p|q}$. Furthermore, if the classic limit, N_C , of N is orientable then above condition is sufficient also in order to say that $N' \in [N]_{\hat{E}_k}$.*

Corollary 3.7. *Let $\hat{E}_k \subseteq \hat{J}_{m|n}^k(W)$ be a quantum super PDE. Let us consider admissible $(p|q)$ -dimensional, $0 \leq p \leq m-1$, $0 \leq q \leq n-1$, integral quantum supermanifolds, with orientable classic limits. Let $N_1 \in [N_2]_{E_k} \in \Omega_{p|q}^{\hat{E}_k}$, then there exists a $(p+1|q+1)$ -dimensional admissible integral quantum supermanifold $V \subset \hat{E}_k$, such that $\partial V = N_1 \sqcup N_2$, where V is without discontinuities iff the integral supernumbers of N_1 and N_2 coincide.*

Above considerations can be generalized to include more sophisticated quantum solutions of quantum super PDEs.

Definition 3.8. *Let $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ be a quantum super PDE and let B be a quantum superalgebra. Let us consider the following chain complex (bigraded bar quantum chain complex of \hat{E}_k): $\{\bar{C}_{\bullet|\bullet}(\hat{E}_k; B), \partial\}$, induced by the \mathbb{Z}_2 -gradation of B on the corresponding bar quantum chain complex of \hat{E}_k , i.e., $\{\bar{C}_{\bullet}(\hat{E}_k; B), \partial\}$.*

(See Refs.[70].) More precisely $\bar{C}_p(\hat{E}_k; B)$ is the free two-sided B -module of formal linear combinations with coefficients in B , $\sum \lambda_i c_i$, where c_i is a singular p -chain $f : \Delta^p \rightarrow \hat{E}_k$, that extends on a neighborhood $U \subset \mathbb{R}^{p+1}$, such that f on U is differentiable and $Tf(\Delta^p) \subset \hat{\mathbf{E}}_{m|n}^k$, where $\hat{\mathbf{E}}_{m|n}^k$ is the Cartan distribution of \hat{E}_k .

Theorem 3.9. [70] *The homology $\bar{H}_{\bullet|\bullet}(\hat{E}_k; B)$ of the bigraded bar quantum chain complex of \hat{E}_k is isomorphic to (closed) bar integral singular $(p|q)$ -bordism groups, with coefficients in B , of \hat{E}_k : ${}^B\bar{\Omega}_{p|q,s}^{\hat{E}_k} \cong \bar{H}_{q|q}(\hat{E}_k; B) \cong (\bar{\Omega}_{p,s}^{\hat{E}_k} \otimes_{\mathbb{K}} B_0) \oplus (\bar{\Omega}_{q,s}^{\hat{E}_k} \otimes_{\mathbb{K}} B_1)$, $p \in \{0, 1, \dots, m-1\}$, $q \in \{0, 1, \dots, n-1\}$. (If $B = \mathbb{K}$ we omit the apex B). If $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ is formally quantum superintegrable and completely superintegrable, and the symbols $\hat{g}_{k+s} \neq 0$, then one has the following canonical isomorphisms: ${}^A\bar{\Omega}_{p|q,s}^{\hat{E}_k} \cong \Omega_{p|q,w}^{\hat{E}_k} \cong \Omega_{p|q,s}^{\hat{E}_k} \cong \Omega_{p|q}(\hat{E}_k)$. Furthermore, the quantum $(p|q)$ -bordism groups $\Omega_{p|q}(\hat{E}_k)$ is an extension of a subgroup of ${}^A\bar{\Omega}_{p|q,s}(W) \cong H_{p|q}(W; A)$, and the integral $(p|q)$ -bordism group $\Omega_{p|q}^{\hat{E}_k}$ is an extension of the quantum $(p|q)$ -bordism group.*

Corollary 3.10. *Let $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ be a quantum super PDE, that is formally superintegrable and completely superintegrable. One has the following isomorphisms: $\Omega_{m-1|n-1,w}^{\hat{E}_k} \cong \Omega_{m-1|n-1}(\hat{E}_k) \cong \Omega_{m-1|n-1,w}^{\hat{E}_{k+h}} \cong \Omega_{m-1|n-1,w}^{\hat{E}_\infty} \cong \Omega_{m-1|n-1,w}(\hat{E}_{k+h}) \cong \Omega_{m-1|n-1}(\hat{E}_\infty) \cong {}^A\bar{\Omega}_{m-1|n-1,s}(W) \cong H_{m-1|n-1}(W; A)$.*

Definition 3.11. *The full space of $(p|q)$ -conservation laws, (or full $(p|q)$ -Hopf superalgebra), of \hat{E}_k is the following one: $\mathbf{H}_{p|q}(E_k) \equiv B^{\Omega_{p|q}^{\hat{E}_k}}$, where $B \equiv \prod_k B_k$. We call full Hopf superalgebra, of \hat{E}_k , the following: $\mathbf{H}_{m-1|n-1}(\hat{E}_\infty) \equiv B^{\Omega_{m-1|n-1}^{\hat{E}_\infty}}$.*

Definition 3.12. *The space of (differential) conservation laws of $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$, is $\mathfrak{Cons}(\hat{E}_k) = \hat{\mathcal{J}}(\hat{E}_\infty)^{m-1|n-1}$.*

Theorem 3.13. [70] *The full $(p|q)$ -Hopf superalgebra of a quantum super PDE $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ has a natural structure of quantum Hopf superalgebra. Quantum Hopf algebras are generalizations of such algebras.*

Proposition 3.14. *The space of conservation laws of \hat{E}_k has a canonical representation in $\mathbf{H}_{m-1|n-1}(\hat{E}_\infty)$.*

Proof. In fact, one has the following homomorphism $j : \mathfrak{Cons}(E_k) \rightarrow \mathbf{H}_{m-1|n-1}(\hat{E}_\infty)$, $j[\alpha]([N]_{\hat{E}_\infty}) = [\alpha], [N]_{\hat{E}_\infty} = \int_{N_C} i^* \alpha \in B$, where $i : N_C \rightarrow N$ is the canonical injection. \square

Theorem 3.15. *Set: $\mathbf{K}_{m-1|n-1,w/(s,w)}^{\hat{E}_k} \equiv B^{K_{n-1,w/(s,w)}^{\hat{E}_k}}$, $\mathbf{K}_{m-1|n-1,w}^{\hat{E}_k} \equiv B^{K_{m-1|n-1,w}^{\hat{E}_k}}$, $\mathbf{K}_{m-1|n-1,s,w}^{\hat{E}_k} \equiv B^{K_{m-1|n-1,(s,w)}^{\hat{E}_k}}$, $\mathbf{K}_{m-1|n-1,s}^{\hat{E}_k} \equiv B^{K_{m-1|n-1,s}^{\hat{E}_k}}$, $\mathbf{H}_{m-1|n-1}(\hat{E}_k) \equiv B^{\Omega_{m-1|n-1}^{\hat{E}_k}}$, $\mathbf{H}_{m-1|n-1,s}(\hat{E}_k) \equiv B^{\Omega_{m-1|n-1,s}^{\hat{E}_k}}$, $\mathbf{H}_{m-1|n-1,w}(\hat{E}_k) \equiv B^{\Omega_{m-1|n-1,w}^{\hat{E}_k}}$. One*

has the following canonical isomorphisms:

$$\begin{aligned}
\mathbf{K}_{m-1|n-1,w/(s,w)}^{\hat{E}_k} &\cong \mathbf{K}_{m-1|n-1,s}^{K^{\hat{E}_k}}; \\
\mathbf{K}_{m-1|n-1,w}^{\hat{E}_k} / \mathbf{K}_{n-1,s,w}^{\hat{E}_k} &\cong \mathbf{K}_{m-1|n-1,w/(s,w)}^{K^{\hat{E}_k}}; \\
\mathbf{H}_{m-1|n-1}(\hat{E}_k) / \mathbf{H}_{m-1|n-1,s}(\hat{E}_k) &\cong \mathbf{K}_{m-1|n-1,s}^{\hat{E}_k}; \\
\mathbf{H}_{m-1|n-1}(\hat{E}_k) / \mathbf{H}_{m-1|n-1,w}(\hat{E}_k) &\cong \mathbf{K}_{m-1|n-1,w}^{\hat{E}_k} \\
&\cong \mathbf{H}_{m-1|n-1,s}(\hat{E}_k) / \mathbf{H}_{m-1|n-1,w}(\hat{E}_k) \cong \mathbf{K}_{m-1|n-1,s,w}^{\hat{E}_k}.
\end{aligned}$$

Proof. The proof is obtained directly by duality of the exact commutative diagram (17). \square

Theorem 3.16. *Under the same hypotheses of Theorem 3.3, one has the following canonical isomorphism: $\mathbf{H}_{m-1|n-1,s}(E_k) \cong \mathbf{H}_{m-1|n-1,w}(\hat{E}_k)$. Furthermore, we can represent differential conservation laws of E_k in $\mathbf{H}_{m-1|n-1,w}(\hat{E}_k)$.*

Proof. Let us note that $\hat{\mathcal{J}}(\hat{E}_k)^{m-1|n-1} \subset \hat{\mathcal{J}}(\hat{E}_\infty)^{m-1|n-1}$. If $j : \mathfrak{Cons}(\hat{E}_k) \rightarrow \mathbf{H}_{m-1|n-1}(\hat{E}_\infty)$, is the canonical representation of the space of the differential conservation laws in the full Hopf superalgebra of \hat{E}_k , (corresponding to the integral bordism groups for regular smooth solutions), it follows that one has also the following canonical representation $j|_{\hat{\mathcal{J}}(\hat{E}_k)^{m-1|n-1}} : \hat{\mathcal{J}}(\hat{E}_k)^{m-1|n-1} \rightarrow \mathbf{H}_{m-1|n-1,s}(\hat{E}_k) \cong \mathbf{H}_{m-1|n-1,w}(\hat{E}_k)$. In fact, for any $N' \in [N]_{\hat{E}_k,s} \in \Omega_{m-1|n-1,s}^{\hat{E}_k} \cong \Omega_{m-1|n-1,w}^{\hat{E}_k}$, one has $\int_{N'} \beta = \int_N \beta$, for any $[\beta] \in \hat{\mathcal{J}}(\hat{E}_k)^{m-1|n-1}$. \square

Theorem 3.17 (Quantum tunnel effects and quantum black holes). *The quantum supergravity equation $\hat{E}_2 \subset J\hat{D}^2(i^*\bar{C})$ admits global solutions having a change of sectional topology (quantum tunnel effects). In general these solutions are not globally representable as second derivative of sections of the fiber bundle $i^*\bar{C} \rightarrow N$. \hat{E}_2 admits solutions that represent evaporations of quantum black holes.*

TABLE 3. Dynamic Equation on macroscopic shell: $\hat{E}_2[i] \subset J\hat{D}^2(i^*\bar{C})$ and Bianchi identity.

Fields equations $(\hat{E}_2[i])$	$(\partial\omega_{ab}^\gamma.L) - \partial_\mu(\partial\omega_{ab}^\mu.L) = 0$ (curvature equation) $(\partial\theta_a^\gamma.L) - \partial_\mu(\partial\theta_a^\mu.L) = 0$ (torsion equation) $(\partial\psi_{\beta i}^\gamma.L) - \partial_\mu(\partial\psi_{\beta i}^\mu.L) = 0$ (gravitino equation) $(\partial A^\gamma.L) - \partial_\mu(\partial A^{\gamma\mu}.L) = 0$ (Maxwell's equation)
Bianchi identity $(B[i])$	$(\partial x_{[\gamma}.R_{\beta\alpha]}^{ab}) + 2\omega_{e[\gamma}^a\omega_{\beta\alpha]}^{eb} = 0$ $(\partial x_{[\gamma}.R_{\beta\omega]}^\alpha) + \omega_{[\gamma}^{\alpha b}R_{\beta\omega]b} + \frac{1}{2}(C\gamma^\alpha)_{\delta\mu}\psi_{j[\gamma}^\delta\rho_{\beta\omega]}^{\mu j} = 0$ $(\partial x_{[\gamma}.\rho_{\omega\alpha]}^{\beta i}) + \frac{1}{2}(\sigma_{ab})_{\delta j}^{\beta i}\omega_{[\gamma}^{ab}\rho_{\omega\alpha]}^{\delta j} = 0$ $(\partial x_{[\gamma}F_{\beta\alpha]}) + \frac{1}{2}C_{\delta\mu}\epsilon_{ij}\psi_{[\gamma}^{\delta i}\rho_{\beta\alpha]}^{\mu j} = 0$
Fields	$R_{\mu\nu}^{ab} = (\partial x_{[\mu}.\omega_{\nu]}^{ab}) + 2\omega_{e[\mu}^a\omega_{\nu]}^{eb}$ (curvature) $R_{\mu\nu}^\alpha = (\partial x_{[\mu}.\theta_{\nu]}^\alpha) + \omega_{\beta[\mu}^\alpha\theta_{\nu]}^\beta + \frac{1}{2}(C\gamma^\alpha)_{\beta\delta}\psi_{j[\mu}^\beta\psi_{\nu]}^{\delta j}$ (torsion) $\rho_{\mu\nu}^{\beta i} = (\partial x_{[\mu}.\psi_{\nu]}^{\beta i}) + \frac{1}{2}(\sigma_{ab})_{\gamma j}^{\beta i}\omega_{[\mu}^{ab}\psi_{\nu]}^{\gamma j}$ (gravitino) $F_{\mu\nu} = (\partial x_{[\mu}.A_{\nu]}) + \frac{1}{2}C_{\beta\gamma}\epsilon_{ij}\psi_{[\mu}^{\beta i}\psi_{\nu]}^{\gamma j}$ (electromagnetic field)

Proof. We shall consider, now, the *quantum* $N = 2$ *superPoincaré group* over a quantum superalgebra $A = A_0 \oplus A_1$, that is a quantum Lie supergroup G having as

quantum Lie superalgebra $\hat{\mathfrak{g}}$ one identified by the following infinitesimal generators: $\{Z_K\}_{1 \leq K \leq 19} \equiv \{J_{\alpha\beta}, P_\alpha, \bar{Z}, Q_{\beta i}\}_{0 \leq \alpha, \beta \leq 3; 1 \leq i \leq 2}$, such that $J_{\alpha\beta} = -J_{\beta\alpha}$, $P_\alpha, \bar{Z} \in \text{Hom}_Z(A_0; \mathfrak{g})$, $Q_{\beta i} \in \text{Hom}_Z(A_1; \mathfrak{g})$. The corresponding nonzero \mathbb{Z}_2 -graded brackets are the following: $[J_{\alpha\beta}, J_{\gamma\delta}] = \eta_{\beta\gamma}J_{\alpha\delta} + \eta_{\alpha\delta}J_{\beta\gamma} - \eta_{\alpha\gamma}J_{\beta\delta} - \eta_{\beta\delta}J_{\alpha\gamma}$, $[P_\alpha, P_\beta] = -8e^2J_{\alpha\beta}$, $[J_{\alpha\beta}, P_\gamma] = \eta_{\beta\gamma}P_\beta - \eta_{\alpha\gamma}P_\beta$, $[J_{\alpha\beta}, Q_{\gamma i}] = (\sigma_{\alpha\beta})_\gamma^{\mu j}Q_{\mu j}$, $[Q_{\beta i}, Q_{\mu j}] = (C\gamma^\alpha)_{\beta\mu}\delta_{ij}P_\alpha + C_{\beta\mu}\epsilon_{ij}\bar{Z}$. Here $C_{\alpha\beta}$ is the antisymmetric charge conjugation matrix, $\sigma_{\beta\mu} = \frac{1}{4}[\gamma_\beta, \gamma_\mu]$, with γ^μ the Dirac matrices. \bar{Z} commutes with all the other ones. One has $\dim G = (d|N_2) = (11|8)$, and we will consider the following principal bundle in the category of quantum supermanifolds: P is a quantum supermanifold of dimension $(15|8)$; M is a quantum supermanifold of dimension $(4|N_1) = (4|0)$, identified, for the sake of simplicity, with a quantum Minkowski space-time. Then a pseudoconnection can be written by means of the following fullquantum differential 1-forms on P : $\gamma\mu^K = \mu_H^K dY^H$, $(\mu_H^K) = (\frac{1}{2}\omega_H^{\alpha\beta}, \theta_H^\mu, A_H, \psi_H^{\alpha j})$. With respect to a section $s : M \rightarrow P$ we get: $(s^*\gamma\mu)^K = \bar{\mu}_\gamma^K dX^\gamma$, $(\bar{\mu}_\gamma^K) = (\frac{1}{2}\bar{\omega}_\gamma^{\alpha\beta}, \bar{\theta}_\gamma^\mu, \bar{A}_\gamma, \bar{\psi}_\gamma^{\alpha j})$, where $\bar{\omega}_\gamma^{\alpha\beta}$ is the usual Levi-Civita connection, $\bar{\theta}_\gamma^\mu$ is the vierbein, \bar{A}_γ is the electromagnetic field and $\bar{\psi}_\gamma^{\alpha j}$ is the usual spin $\frac{3}{2}$ field. The blow up structure: $\pi^*\hat{C}(P) \hookrightarrow \text{Hom}_Z(TP; \mathfrak{g})$ implies that we can identify our fields with sections $\gamma\mu$ of the fiber bundle $\bar{\pi} : \bar{C} \equiv \text{Hom}_Z(TM; \mathfrak{g}) \rightarrow M$. ($\hat{C}(P) \cong J\hat{D}(P)/G$ is the fiber bundle, over M , of principal quantum connections on the G -principal fiber bundle $\pi : P \rightarrow M$.) The corresponding curvatures can be written in the form: $\gamma R_{\beta\alpha}^K = (\partial x_\beta \mu_\alpha^K) + C_{IJ}^K[\mu_\beta^I, \mu_\alpha^J]_+$. The local expression of the *dynamic equation*, $\hat{E}_2[i] \subset J\hat{D}^2(i^*\bar{C})$, evaluated on a macroscopic shell, i.e., an embedding $i : N \rightarrow M$, of a globally hyperbolic, p -connected manifold N , $0 \leq p \leq 3$, is given by the quantum super PDE reported in Tab.3.1, where $L : J\hat{D}(\underline{E}) \rightarrow \hat{A}$ is a quantum Lagrangian function. Possible Lagrangian densities are polynomial in the curvature, (see example below), hence we can assume that they give formally quantum superintegrable, and completely quantum superintegrable, quantum super PDE's. Then, assuming that $\hat{E}_2[i]$ is formally integrable and completely superintegrable, the integral bordism groups of $\hat{E}_2[i]$ and its fullquantum p -Hopf superalgebras, can be calculated. More precisely, we use the fact that $\hat{C}(P) \rightarrow M$ is a contractible fiber bundle of dimension $(4|0, 44|32)$

over the quantum superalgebra $A \times \hat{A} = (A_0 \times A_1) \times \hat{A}_0(A) \times \hat{A}_1(A)$, and that N is topologically trivial. In fact, we can apply Theorem 3.9 and Corollary 3.10, to obtain the quantum and integral bordism groups of $\hat{E}_2[i]$: $\Omega_{p,s}^{\hat{E}_2[i]} \cong \Omega_{p,w}^{\hat{E}_2[i]+\infty} \cong 0$, for $p = 1, 2, 3$ and $\Omega_{0,s}^{\hat{E}_2[i]} \cong \Omega_{0,w}^{\hat{E}_2[i]+\infty} \cong A$. Therefore, we have that 1-dimensional admissible integral closed quantum submanifolds contained into $\hat{E}_2[i]$, (*admissible quantum closed strings*), can propagate and interact between them by means of 2-dimensional admissible integral quantum manifolds contained into $\hat{J}_4^2(i^*\bar{C})$, or by means of 2-dimensional admissible integral quantum manifolds contained into $\hat{E}_2[i]$, in such a way to generate (quantum) tunnel effects. Finally, as a consequence of the triviality of the 3-dimensional integral bordism groups, we get the existence of global quantum solutions of such equations.

Let us now see that theorem can be proved by using surgery techniques and taking into account that for the 3-dimensional integral bordism group of \hat{E}_k one has $\Omega_{3,s}^{\hat{E}_k} = 0 = \Omega_{3,w}^{(\hat{E}_2)+\infty}$. In fact a boundary value problem for $\hat{E}_k[i]$ can be directly implemented in the manifold $\hat{E}_k[i] \subset J\hat{D}^2(i^*\bar{C}) \subset \hat{J}_4^2(i^*\bar{C})$ by requiring that a

3-dimensional compact space-like (for some $t = t_0$), admissible integral manifold $B \subset \hat{E}_k[i]$ propagates in $\hat{E}_k[i]$ in such a way that the boundary ∂B describes a fixed 3-dimensional time-like integral manifold $Y \subset \hat{E}_k[i]$. (We shall require that the boundary ∂B of B is orientable.) Y is not, in general, a closed (smooth) manifold. However, we can solder Y with two other compact 3-dimensional integral manifolds X_i , $i = 1, 2$, in such a way that the result is a closed 3-dimensional (smooth) integral manifold $Z \subset \hat{E}_k[i]$. More precisely, we can take $X_1 = B$ so that $\tilde{Z} \equiv X_1 \cup_{\partial B} Y$ is a 3-dimensional compact integral manifold such that $\partial \tilde{Z} \equiv C$ is a 2-dimensional space-like integral manifold. We can assume that C is an orientable manifold. Then, from the triviality of the integral bordism group, it follows that $\partial X_2 = C$, for some space-like compact 3-dimensional integral manifold $X_2 \subset \hat{E}_k[i]$. Set $Z \equiv \tilde{Z} \cup_C X_2$. Therefore, one has $Z = X_1 \cup_{\partial B} Y \cup_C X_2$. Then, again from the triviality of the integral bordism group, it follows also that there exists a 4-dimensional integral (smooth) manifold $V \subset \hat{E}_k[i]$ such that $\partial V = Z$. Hence the integral manifold V is a solution of our boundary value problem between the times t_0 and t_1 , where t_0 and t_1 are the times corresponding to the boundaries where are soldered X_i , $i = 1, 2$ to Y . Now, this process can be extended for any $t_2 > t_1$. So we are able to find (smooth) solutions for any $t > t_0$, hence (smooth) solutions for any $t > t_0$, therefore, global (smooth) solutions. Remark that in order to assure the smoothness of the global solution so built it is enough to develop such construction in the infinity prolongation $\hat{E}_k[i]_{+\infty}$ of $\hat{E}_k[i]$. Finally note that in the set of solutions of $\hat{E}_k[i]$ there are ones that have change of sectional topology. In fact the 3-dimensional integral bordism groups are trivial: $\Omega_{3,s}^{\hat{E}_k[i]} = 0 = \Omega_{3,w}^{\hat{E}_k[i]_{+\infty}}$. Let us, now, consider the dynamics of a quantum black-hole. In order to obtain such solutions we must have a Cauchy integral data with a geometric black hole B embedded in a compact 3-dimensional integral manifold N , $B \subset N$, such that its boundary ∂N propagates with a fixed flow. Then a solution, with quantum tunnel effect of such boundary problem, can describe a evaporation process of such black hole. Above results assure the existence of such solutions and a way to build them. In order to represent such results with respect to a quantum relativistic observer, let us consider the space of observed quantum integral conservation laws.

Definition 3.18.

$$\begin{cases} \hat{\mathcal{J}}(\hat{E}_k[i]) & \equiv \oplus_{q \geq 0} \frac{\hat{\Omega}^q(\hat{E}_k[i]) \cap d^{-1}(C\hat{\Omega}^{q+1}(\hat{E}_k[i]))}{d\hat{\Omega}^{q-1}(\hat{E}_k[i]) \oplus \{C\hat{\Omega}^q(\hat{E}_k[i]) \cap d^{-1}(C\hat{\Omega}^{q+1}(\hat{E}_k[i]))\}} \\ & \equiv \oplus_{q \geq 0} \hat{\mathcal{J}}(\hat{E}_k[i])^q. \end{cases}$$

Here $C\hat{\Omega}^q(\hat{E}_k[i])$ denotes the space of all Cartan quantum q -forms on $\hat{E}_k[i]$. (See also [70].) Then we define quantum integral characteristic supernumbers of N , with $[N] \in \hat{\Omega}_q^{\hat{E}_k[i]}$, the numbers $\hat{i}[N] \equiv \langle [N], [\alpha] \rangle \in B$, for all $[\alpha] \in \hat{\mathcal{J}}(\hat{E}_k[i])^q$.

One has the following lemma.

Lemma 3.19. [70] *Let us assume that $\hat{\mathcal{J}}(\hat{E}_k[i])^q \neq 0$. One has a natural homomorphism: $\hat{j}_q : \hat{\Omega}_q^{\hat{E}_k[i]} \rightarrow \text{Hom}_A(\hat{\mathcal{J}}(\hat{E}_k[i])^q; A)$, $[N] \mapsto \hat{j}_q([N])$, $\hat{j}_q([N])([\alpha]) = \int_N \alpha \equiv \langle [N], [\alpha] \rangle$. Then, a necessary condition that $N' \in [N] \in \hat{\Omega}_q^{\hat{E}_k[i]}$ is the following: $\hat{i}[N] = \hat{i}[N']$, $\forall [\alpha] \in \hat{\mathcal{J}}(\hat{E}_k[i])^q$. Furthermore, if N is orientable then above condition is sufficient also in order to say that $N' \in [N]$.*

Therefore a quantum evaporation black-hole process can be described by means of quantum smooth integral manifolds, and therefore for such a process "conservation laws" are not destroyed. By the way, as we can have also weak solutions around a quantum black-hole, we can assume also that interactions with such objects could be described by means of weak-solutions, like shock-waves. Therefore we shall more precisely talk of *weak quantum black-holes* and *non-weak quantum black-holes*, according if they are described respectively by means of weak solutions, or non-weak solutions. As a by-product we get that all the quantum integral characteristic supernumbers are conserved through a non-weak quantum evaporating black-hole. \square

Let us conclude this section by considering the following theorem that gives an important quantum conservation law for what we shall develop in part II.

Theorem 3.20 (Observed quantum Hamiltonian as quantum conservation law of $(\widehat{YM})[i]$). *The observed quantum super Yang-Mills PDE $(\widehat{YM})[i]$ admits a quantum conservation law that can be identified with the observed Hamiltonian on any space-like section of any observed solution of $(\widehat{YM})[i]$.*

Proof. Let us recall that the observed quantum Hamiltonian of the observed quantum super Yang-Mills equation can be written in the form reported in (19). (See [67, 70, 75].)

$$(19) \quad H = (\partial y_\alpha^\beta . L) y_\beta^\alpha - L$$

where, for sake of simplicity, we have put $\{x^\beta, y^\alpha\}$ the coordinates on the observed configuration bundle for the observed quantum super Yang-Mills equation $(\widehat{YM})[i]$. Since equation $(\widehat{YM})[i]$ is invariant for time translations, we get that the quantum differential form (20), is a quantum conservation laws of $(\widehat{YM})[i]$, in the sense of Definition 3.18.⁹

$$(20) \quad \begin{cases} \omega_H & \equiv \partial x_0 [(Ds)^* \Theta + (Ds)^* (\partial x_0) \Theta] \\ \Theta & = (-1)^{\beta+1} (\partial y_\alpha^\beta . L) \otimes dy^\alpha \wedge dx^0 \wedge \cdots \wedge \widehat{dx^\beta} \wedge \cdots \wedge dx^3 + H \otimes dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{cases}$$

Here $\widehat{dx^\beta}$ denotes absent with $\beta \in \{0, 1, 2, 3\}$. In fact, for any solution s of $(\widehat{YM})[i]$, we get $d\omega_H = 0$. An explicit calculation of ω_H in quantum coordinates gives the expression reported in (21). More explicitly one has the following.

$$(21) \quad \begin{cases} \omega_H|V & = -[(\partial y_\alpha^\beta . L) y_\beta^\alpha - H \circ Ds] \otimes dx^1 \wedge dx^2 \wedge dx^3 \\ & + (-1)^{\beta+1} (\partial y_\alpha^\beta . L) y_0^\alpha \otimes dx^0 \wedge \cdots \wedge \widehat{dx^\beta} \wedge \cdots \wedge dx^3 \\ & + (\partial y_\alpha^j . L) y_j^\alpha \otimes dx^1 \wedge dx^2 \wedge dx^3 \end{cases}$$

Here $\widehat{dx^\beta}$ denotes absent with $\beta \in \{1, 2, 3\}$. The restriction of $\omega_H|V$ on any compact 3-dimensional space-like section $\sigma_t \subset V$, $t = \text{const}$, of the solution V of $(\widehat{YM})[i]$, coincides with the restriction of H on σ_t . More precisely we get the formula (22).

$$(22) \quad \omega_H|_{\sigma_t} = H|_{\sigma_t} = [(\partial y_\alpha^j . L) y_j^\alpha - L] \otimes dx^1 \wedge dx^2 \wedge dx^3.$$

⁹For complementary information of variational calculus on PDEs and quantum super PDEs see [69, 75].

Then the evaluation of ω_H on σ_t , gives a time dependent A -valued function $H[i|t]$, defined in (23)

$$(23) \quad \begin{cases} H[i|t] & \equiv \int_{\sigma_t} \omega_H|V \\ & = \int_{\sigma_t} [(\partial y_\alpha^j \cdot L) y_j^\alpha - L] \otimes dx^1 \wedge dx^2 \wedge dx^3 \\ & = \int_{\sigma_t} H|_{\sigma_t} \otimes dx^1 \wedge dx^2 \wedge dx^3 \in A. \end{cases}$$

This function does not necessitate to be constant. In fact, we get

$$0 = \int_V d\omega_H|V = \int_{\partial V} \omega_H|_{\partial V} = \int_{N_0} \omega_H|_{N_0} - \int_{N_1} \omega_H|_{N_1} + \int_P \omega_H|_P.$$

We get

$$(24) \quad H[i|t_0] - H[i|t_1] = - \int_P \omega_H|_P \in A.$$

Therefore, $H[i|t_0] = H[i|t_1]$ iff $\int_P \omega_H|_P = 0 \in A$. In general this last condition is not verified. \square

Definition 3.21 (Lost quantum energy of an observed quantum nonlinear propagator of $(\widehat{YM})[i]$). *Given an observed quantum nonlinear propagator V of $(\widehat{YM})[i]$, such that $\partial V = N_0 \sqcup P \sqcup N_1$, where $N_i, i = 0, 1$, are 3-dimensional space-like admissible Cauchy data of $(\widehat{YM})[i]$, and P is a suitable time-like 3-dimensional integral manifold with $\partial P = \partial N_0 \sqcup \partial N_1$, we define $L[V] \equiv \int_P \omega_H|_P$ lost quantum energy of V .*

Corollary 3.22. *For any observed quantum nonlinear propagator V of $(\widehat{YM})[i]$, such that $\partial V = N_0 \sqcup P \sqcup N_1$, where $N_i, i = 0, 1$, are 3-dimensional space-like admissible Cauchy data of $(\widehat{YM})[i]$, and P is a suitable time-like 3-dimensional integral manifold with $\partial P = \partial N_0 \sqcup \partial N_1$, equation (25) holds.*

$$(25) \quad H[i|t_0] = H[i|t_1] \bmod L[V] \in A$$

where t_0 (resp. t_1), is the time of N_0 (resp. N_1).

Corollary 3.22 gives a precise meaning to the phenomenological statement that with respect to an observer a quantum process must conserve the total mass-energy.

Example 3.23 (Steady-states in $(\widehat{YM})[i]$). *Steady state solutions of $(\widehat{YM})[i]$ are ones where the partial derivative with respect to the time of the observed fundamental field $\tilde{\mu}_\alpha^K$ is zero: $\tilde{\mu}_{\alpha 0}^K = 0$. Then with the symbols used in the proof of Theorem 3.20, we get that the lost quantum energy $L[V] \in A$ of a quantum nonlinear propagator for such a solution must necessarily be zero. In fact, we get*

$$L[V] \equiv \int_P \omega_H|_P = \int_P [(-1)^{\beta+1} (\partial y_\alpha^\beta \cdot L) y_0^\alpha] \otimes dx^0 \wedge \cdots \wedge \widehat{dx^\beta} \wedge \cdots \wedge dx^3 = 0 \in A$$

since $y_0^\alpha = 0$. Therefore, in the steady state quantum nonlinear propagators holds the strict conservation of the observed quantum energy: $H[i|t_0] = H[i|t_1] \in A$.

Example 3.24 (Strong reactions with jet quenching in $(\widehat{YM})[i]$). *Quantum nonlinear propagators with lost quantum energy can encode strong reactions between ultra-relativistic heavy-ion collisions, where interactions between the high-momentum parton and the hot, dense medium produced in the collisions, lead to energy loss. This phenomenon is called jet quenching.*¹⁰

4. QUANTUM HYPERCOMPLEX EXOTIC SUPER PDE'S

In this section we consider PDE's in the category $\mathfrak{Q}_{hyper,S}$ of quantum hypercomplex supermanifolds, as defined in [83], and focus our attention on "quantum exotic super PDE's", i.e., quantum super PDE's where we can embed "quantum exotic superspheres". For such Cauchy data we will generalize our previous results on quantum exotic PDE's [83]. Such "exotic" boundary value problems are of particular interest in strong reactions encoding quantum processes occurring in high energy physics, as we will prove in part II.

Remark 4.1. *Let us remark that in the other sections of this paper we refer to the category \mathfrak{Q}_S instead that $\mathfrak{Q}_{hyper,S}$. This is made for three reasons. The first is for convenience, since quantum micro-worlds can be encoded in \mathfrak{Q}_S . The second is that one can directly generalize intrinsic results obtained in the geometry of PDEs in the category \mathfrak{Q}_S to similar ones in the category $\mathfrak{Q}_{hyper,S}$, as we have proved in [83], and we will also see in this section. The third reason is that nonassociative algebras, as arise for example in some quantum hypercomplex algebras, can be considered as subsets of their enveloping algebras. These last being necessarily associative, give a general criterion to encode nonassociative algebras in larger associative frameworks.*

Definition 4.2 (Quantum homotopy $(m|n)$ -supersphere). *We call quantum homotopy $(m|n)$ -supersphere (with respect to a quantum hypercomplex superalgebra A) a smooth, compact, closed $(m|n)$ -dimensional quantum supermanifold $M \equiv \hat{\Sigma}^{m|n}$, that is homotopy equivalent to the $(m|n)$ -dimensional quantum supersphere $\hat{S}^{m|n}$, with classic regular structure $\pi_C : M \rightarrow M_C$, where M_C is a homotopy m -sphere, and such that the homotopy equivalence between M and $\hat{S}^{m|n}$ is realized by a commutative diagram (26).*

$$(26) \quad \begin{array}{ccccccc} M & \xrightarrow{f} & \hat{S}^{m|n} & \equiv & A^{m|n} \cup \{\infty\} & \equiv & (A_0^m \times A_1^n) \cup \{\infty\} \\ \pi_C \downarrow & & \downarrow \pi_C & & \downarrow & & \downarrow \\ M_C & \xrightarrow{f_C} & S^m & \equiv & \mathbb{K}^m \cup \{\infty_C\} & \equiv & \mathbb{K}^m \cup \{\infty_C\} \end{array}$$

$\mathbb{K} = \mathbb{R}, \mathbb{C}$ and π_C is induced by the canonical mappings $c : A \rightarrow \mathbb{K}$ and $\infty \mapsto \infty_C$.

Remark 4.3. *Let $\hat{\Sigma}_1^{m|n}$ and $\hat{\Sigma}_2^{m|n}$ be two quantum diffeomorphic, quantum homotopy $(m|n)$ -superspheres: $\hat{\Sigma}_1^{m|n} \cong \hat{\Sigma}_2^{m|n}$. Then the corresponding classic limits $\hat{\Sigma}_{1,C}^{m|n}$ and $\hat{\Sigma}_{2,C}^{m|n}$ are diffeomorphic too: $\hat{\Sigma}_{1,C}^{m|n} \cong \hat{\Sigma}_{2,C}^{m|n}$. This remark is the natural consequence of the fact that quantum diffeomorphisms here considered respect the fiber bundle structures of quantum homotopy $(m|n)$ -superspheres with*

¹⁰Parton model was proposed by R. Feynmann in 1969 for high-energy hadron collisions and actually usually referred as quark-gluon model. Nowadays there exist experimental evidences for the quenching phenomenon. (See, e.g., [1] and CERN-Press Releases reported therein.)

respect their classic limits: $\pi_C : \hat{\Sigma}^{m|n} \rightarrow \hat{\Sigma}_C^{m|n}$. Therefore quantum diffeomorphisms between quantum homotopy $(m|n)$ -superspheres are characterized by a couple $(f, f_C) : (\hat{\Sigma}_1^{m|n}, \hat{\Sigma}_{1,C}^{m|n}) \rightarrow (\hat{\Sigma}_2^{m|n}, \hat{\Sigma}_{2,C}^{m|n})$ of mappings related by the commutative diagram in (27).

$$(27) \quad \begin{array}{ccc} \hat{\Sigma}_1^{m|n} & \xrightarrow{f} & \hat{\Sigma}_2^{m|n} \\ \pi_{1,C} \downarrow & & \downarrow \pi_{2,C} \\ \hat{\Sigma}_{1,C}^{m|n} & \xrightarrow{f_C} & \hat{\Sigma}_{2,C}^{m|n} \end{array}$$

There f is a quantum diffeomorphism between quantum supermanifolds and f_C is a diffeomorphism between manifolds. Note that such diffeomorphisms of quantum homotopy $(m|n)$ -superspheres allow to recognize that $\hat{\Sigma}_1^{m|n}$ has also $\hat{\Sigma}_{2,C}^{m|n}$ as classic limit, other than $\hat{\Sigma}_{1,C}^{m|n}$. (See commutative diagram in (28).)

$$(28) \quad \begin{array}{ccccc} & & \hat{\Sigma}_1^{m|n} & & \\ & \nearrow f^{-1} & & \searrow f & \\ \hat{\Sigma}_2^{m|n} & \xrightarrow{1_{\hat{\Sigma}_2^{m|n}}} & \hat{\Sigma}_2^{m|n} & & \\ \pi'_{2,C} \downarrow & \nearrow \pi_{1,C} & & \searrow \pi'_{1,C} & \downarrow \pi_{2,C} \\ \hat{\Sigma}_{1,C}^{m|n} & \xrightarrow{f_C} & \hat{\Sigma}_{2,C}^{m|n} & & \end{array}$$

This clarifies that the classic limit of a quantum homotopy $(m|n)$ -supersphere is unique up to diffeomorphisms.

Let us also emphasize that (co)homology properties of quantum homotopy $(m|n)$ -superspheres are related to the ones of m -spheres, since we here consider classic regular objects only.

Lemma 4.4. In (29) are reported the cohomology spaces for quantum homotopy $(m|n)$ -superspheres.

$$(29) \quad H^p(\hat{\Sigma}^{m|n}; \mathbb{Z}) \cong H^p(\hat{S}^{m|n}; \mathbb{Z}) \cong H^p(S^m; \mathbb{Z}) = \begin{cases} 0 & p \neq 0, m \\ \mathbb{Z} & p = 0, m. \end{cases}$$

Proof. Let us first calculate the homology groups in integer coefficients \mathbb{Z} , of quantum $(m|n)$ -superspheres. In (30) are reported the homology spaces for quantum $m \neq 0$.

$$(30) \quad H_p(\hat{S}^{m|n}; \mathbb{Z}) \cong H_p(S^m; \mathbb{Z}) = \begin{cases} 0 & p \neq 0, m \\ \mathbb{Z} & p = 0, m. \end{cases}$$

Furthermore, for $m = 0$ we get

$$H_p(\hat{S}^{0|n}; \mathbb{Z}) \cong H_p(S^0; \mathbb{Z}) = \begin{cases} 0 & p \neq 0 \\ \mathbb{Z} \oplus \mathbb{Z} & p = 0. \end{cases}$$

Above formulas can be obtained by the reduced Mayer-Vietoris sequence applied to the triad $(\hat{S}^{m|n}, \hat{D}_+^{m|n}, \hat{D}_-^{m|n})$ since we can write $\hat{S}^{m|n} = \hat{D}_+^{m|n} \cup \hat{D}_-^{m|n}$, where $\hat{D}_+^{m|n}$

and $\hat{D}_-^{m|n}$ are respectively the north quantum $(m|n)$ -superdisk and south quantum $(m|n)$ -superdisk that cover $\hat{S}^{m|n}$. Taking into account that $\hat{D}_+^{m|n} \cap \hat{D}_-^{m|n} = \hat{S}^{m-1|n-1}$, we get the long exact sequence (31).

$$\begin{array}{ccccccc}
 (31) & \cdots & \longrightarrow & \tilde{H}_p(\hat{S}^{m-1|n-1}; \mathbb{Z}) & \longrightarrow & \tilde{H}_p(\hat{D}_+^{m|n}; \mathbb{Z}) \oplus \tilde{H}_p(\hat{D}_-^{m|n}; \mathbb{Z}) & \longrightarrow & \tilde{H}_p(\hat{S}^{m|n}; \mathbb{Z}) \\
 & & & & & & & \downarrow \partial \\
 & & & \tilde{H}_{p-1}(\hat{S}^{m|n}; \mathbb{Z}) & \longleftarrow & \tilde{H}_{p-1}(\hat{D}_+^{m|n}; \mathbb{Z}) \oplus \tilde{H}_{p-1}(\hat{D}_-^{m|n}; \mathbb{Z}) & \longleftarrow & \tilde{H}_{p-1}(\hat{S}^{m-1|n-1}; \mathbb{Z}) \\
 & & & \downarrow \partial \\
 & & & \tilde{H}_{p-2}(\hat{S}^{m-1|n-1}; \mathbb{Z}) & \longrightarrow & \tilde{H}_{p-2}(\hat{D}_+^{m|n}; \mathbb{Z}) \oplus \tilde{H}_{p-2}(\hat{D}_-^{m|n}; \mathbb{Z}) & \longrightarrow & \tilde{H}_{p-2}(\hat{S}^{m|n}; \mathbb{Z}) \\
 & & & & & & & \downarrow \partial \\
 & & & \vdots & & & & \\
 & & & \downarrow \partial \\
 & & & \tilde{H}_0(\hat{S}^{m-1|n-1}; \mathbb{Z}) & \longrightarrow & \tilde{H}_0(\hat{D}_+^{m|n}; \mathbb{Z}) \oplus \tilde{H}_0(\hat{D}_-^{m|n}; \mathbb{Z}) & \longrightarrow & \tilde{H}_0(\hat{S}^{m|n}; \mathbb{Z}) \longrightarrow 0.
 \end{array}$$

Taking into account that $\tilde{H}_0(\hat{D}_-^{m|n}; \mathbb{Z}) = 0$ we get $\tilde{H}_p(\hat{S}^{m|n}; \mathbb{Z}) \cong \tilde{H}_{p-1}(\hat{S}^{m-1|n-1}; \mathbb{Z})$ and $\tilde{H}_0(\hat{S}^{m|n}; \mathbb{Z}) \cong 0$. Therefore, we get

$$(32) \quad \tilde{H}_p(\hat{S}^{m|n}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } p = m \\ 0 & \text{if } p \neq m \end{cases} \Rightarrow \begin{cases} H_p(\hat{S}^{0|n}; \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } m = 0 \\ 0 & \text{if } p \neq 0 \end{cases} \\ H_p(\hat{S}^{m|n}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, m \\ 0 & \text{if } p \neq 0, m. \end{cases} \end{cases}$$

Therefore we get formulas (30). To conclude the proof we shall consider that $H^p(\hat{S}^{m|n}; \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(H_p(\hat{S}^{m|n}; \mathbb{Z}); \mathbb{Z})$. Furthermore, quantum homotopy $(m|n)$ -superspheres have same (co)homology of quantum superspheres since are homotopy equivalent to these last ones. \square

Lemma 4.5. *The quantum Euler characteristic numbers for quantum homotopy $(m|n)$ -superspheres are reported in (33). These coincide with the corresponding quantum Euler characteristic numbers of quantum $(m|n)$ -superspheres and with the Euler characteristic numbers of usual m -spheres. Furthermore they are the same of the corresponding total quantum Euler characteristic numbers. See in (33).*

$$(33) \quad \begin{cases} \hat{\chi}(\hat{\Sigma}^{m|n}) &= \hat{\chi}(\hat{S}^{m|n}) = \chi(S^m) = (-1)^0 \beta_0 + (-1)^m \beta_m = 1 + (-1)^m \\ &= \begin{cases} 0 & m = \text{odd} \\ 2 & m = \text{even.} \end{cases} \\ &= {}^{Tot}\hat{\chi}(\hat{\Sigma}^{m|n}) = {}^{Tot}\hat{\chi}(\hat{S}^{m|n}). \end{cases}$$

Proof. We have considered that $\hat{S}^{m|n}$ admits the following quantum-supercell decomposition: $\hat{S}^{m|n} = \hat{e}^{m|n} \cup \hat{e}^{0|0}$, where $\hat{e}^{m|n} = \hat{D}^{m|n}$ is a $(m|n)$ -dimensional quantum supercell, with respect to the quantum superalgebra A , and $\hat{e}^{0|0} = \hat{D}^{0|0}$ is the $(0|0)$ -dimensional quantum supercell with respect to A . Therefore we can consider the quantum homological Euler characteristic $\hat{\chi}(\hat{S}^{m|n})$ of $\hat{S}^{m|n}$, given by formulas

(34).

$$\begin{aligned}
\hat{\chi}(\hat{S}^{m|n}) &= (-1)^0 \dim_A H_0(\hat{S}^{m|n}; A) + (-1)^m \dim_A H_m(\hat{S}^{m|n}; A) \\
&= (-1)^0 \dim_A A + (-1)^m \dim_A A \\
(34) \quad &= 1 + (-1)^m \\
&= \begin{cases} 0 & m = \text{odd} \\ 2 & m = \text{even.} \end{cases}
\end{aligned}$$

So the homological quantum Euler characteristic of the quantum $(m|n)$ -supersphere is the same of the homological Euler characteristic of the usual m -sphere. Furthermore, since quantum homotopy $(m|n)$ -superspheres are homotopy equivalent to quantum $(m|n)$ -superspheres, it follows that the quantum Euler characteristic of a quantum homotopy $(m|n)$ -supersphere is equal to the one of $\hat{S}^{m|n}$. Moreover, also the total quantum Euler characteristic numbers for quantum (homotopy) $(m|n)$ -superspheres coincide with the ones of S^m . In fact, we can consider the *quantum total-homological Euler characteristic* $^{Tot}\hat{\chi}(\hat{S}^{m|n})$ of $\hat{S}^{m|n}$, given by formulas (35).

$$\begin{aligned}
^{Tot}\hat{\chi}(\hat{S}^{m|n}) &= \sum_{p \geq 0} (-1)^p \dim_A ^{Tot}H_p(\hat{S}^{m|n}; A) \\
&= \sum_{p \geq 0} (-1)^p \dim_A [\bigoplus_{r+s=p} H_{r|s}(\hat{S}^{m|n}; A)] \\
(35) \quad &= \sum_{p \geq 0} (-1)^p \dim_A [\bigoplus_{r+s=p} (H_r(\hat{S}^{m|n}; A_0) \oplus H_s(\hat{S}^{m|n}; A_1))] \\
&= \sum_{p \geq 0} (-1)^p \dim_A [\bigoplus_{r+s=p} (H_r(\hat{S}^{m|n}; \mathbb{Z}) \otimes_{\mathbb{Z}} A_0) \oplus H_s(\hat{S}^{m|n}; \mathbb{Z}) \otimes_{\mathbb{Z}} A_1)] \\
&= \sum_{p \geq 0} (-1)^p \dim_A [\bigoplus_{r+s=p} (H_r(S^m; \mathbb{Z}) \otimes_{\mathbb{Z}} A_0) \oplus H_s(S^m; \mathbb{Z}) \otimes_{\mathbb{Z}} A_1)] \\
&= 1 + (-1)^m.
\end{aligned}$$

So also the total homological quantum Euler characteristic of the quantum $(m|n)$ -supersphere is the same of the homological Euler characteristic of the usual m -sphere. Furthermore, since quantum homotopy $(m|n)$ -superspheres are homotopy equivalent to quantum $(m|n)$ -superspheres, it follows that the total quantum Euler characteristic of a quantum homotopy $(m|n)$ -supersphere is equal to the one of $\hat{S}^{m|n}$. \square

Theorem 4.6 (Generalized Poincaré conjecture in the category $\mathfrak{Q}_{hyper,S}$). *Let A be a hypercomplex quantum superalgebra with center $Z(A)$ a Noetherian \mathbb{K} -algebra, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let M be a classic regular, closed compact quantum supermanifold of dimension $(m|n)$, in the category $\mathfrak{Q}_{hyper,S}$, homotopy equivalent to $\hat{S}^{m|n}$, (hence this last is the quantum super CW-substitute of M). Then we get that $M \approx \hat{S}^{m|n}$, i.e., M is also homeomorphic to $\hat{S}^{m|n}$, and $M_C \approx S^m$, i.e., the classic limit M_C of M is homeomorphic to the classic limit S^m of $\hat{S}^{m|n}$.*

Proof. In [75] we have proved the generalized quantum Poincaré conjecture in the category \mathfrak{Q}_S of quantum supermanifolds, by considering the quantum Ricci flow PDE just in the category \mathfrak{Q}_S . Then, by using similar arguments to prove Theorem 3.10 and Theorem 3.11 in [83], we can state that generalized quantum Poincaré conjecture works also in the category $\mathfrak{Q}_{hyper,S}$ of quantum hypercomplex supermanifolds. Therefore the quantum $(m|n)$ -supersphere $\hat{S}^{m|n}$, considered the quantum super CW-substitute of any quantum homotopy $(m|n)$ -supersphere $\hat{\Sigma}^{m|n}$, is just homeomorphic to this last one: $\hat{\Sigma}^{m|n} \approx \hat{S}^{m|n}$. \square

Theorem 4.7. *Let $\hat{\Theta}_{m|n}$ be the set of equivalence classes of quantum diffeomorphic quantum homotopy $(m|n)$ -superspheres over a quantum (hypercomplex) superalgebra A (and with Noetherian center $Z(A)$).¹¹ In $\hat{\Theta}_{m|n}$ it is defined an additive*

¹¹Quantum diffeomorphisms are meant in the sense specified in Remark 4.3.

commutative and associative composition map such that $[\hat{S}^{m|n}]$ is the zero of the composition. Then one has the exact commutative diagram reported in (36).

$$(36) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \hat{\Upsilon}_{m|n} & \hookrightarrow & \hat{\Theta}_{m|n} & \xrightarrow{j_C} & \Theta_m \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & \hat{\Theta}_{m|n}/\hat{\Upsilon}_{m|n} & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

where Θ_m is the set of equivalence classes for diffeomorphic homotopy m -spheres and j_C is the canonical mapping $j_C : [\hat{\Sigma}^{m|n}] \mapsto [\hat{\Sigma}_C^{m|n}]$. One has the canonical isomorphisms:

$$(37) \quad \mathbb{Z} \bigotimes_{\hat{\Upsilon}_{m|n}} \mathbb{Z} \hat{\Theta}_{m|n} \cong \mathbb{Z} \Theta_m, \text{ as right } \hat{\Theta}_{m|n}\text{-modules.}$$

Proof. After above Remark 4.3 we can state that the mapping j_C is surjective. In other words we can write

$$\hat{\Theta}_{m|n} = \bigcup_{[\hat{\Sigma}_C^{m|n}] \in \Theta_m} (\hat{\Theta}_{m|n})_{[\hat{\Sigma}_C^{m|n}]}.$$

The fiber $(\hat{\Theta}_{m|n})_{[\hat{\Sigma}_C^{m|n}]}$ is given by all classes $[\hat{\Sigma}^{m|n}]$ such their classic limits are diffeomorphic, hence belong to the same class in Θ_m . Furthermore one has $\ker(j_C) = j_C^{-1}([S^m]) \equiv \hat{\Upsilon}_{m|n} \subset \hat{\Theta}_{m|n}$. Therefore, we can state that $\hat{\Theta}_{m|n}$ is an extension of Θ_m by $\hat{\Upsilon}_{m|n}$. Such extensions are classified by $H^2(\Theta_m; \hat{\Upsilon}_{m|n})$.¹²

TABLE 4. Homology of finite cyclic group \mathbb{Z}_i of order i .

r	$H_r(\mathbb{Z}_i; \mathbb{Z})$
0	\mathbb{Z}
r odd	\mathbb{Z}_i
$r > 0$ even	0

The composition map in $\hat{\Theta}_{m|n}$ is defined by *quantum fibered connected sum*, i.e., a connected sum on quantum supermanifolds that respects the connected sum on their corresponding classic limits. More precisely let $M \rightarrow M_C$ and $N \rightarrow N_C$ be connected $(m|n)$ -dimensional classic regular quantum supermanifolds. We define quantum fibered connected sum of M and N the classic regular $(m|n)$ -dimensional quantum supermanifold $M \# N \rightarrow M_C \# N_C$, where

$$(38) \quad \begin{cases} M \# N &= (M \setminus \hat{D}^{m|n}) \cup (\hat{S}^{m-1|n-1} \times \hat{D}^{1|1}) \cup (N \setminus \hat{D}^{m|n}) \\ M_C \# N_C &= (M_C \setminus D^m) \cup (S^{m-1} \times D^1) \cup (N_C \setminus D^m). \end{cases}$$

¹²In Tab. 4 are reported useful formulas to explicitly calculate these groups.

Then the additive composition law is $+$: $\hat{\Theta}_{m|n} \times \hat{\Theta}_{m|n} \rightarrow \hat{\Theta}_{m|n}$, $[M] + [N] = [M \# N]$. $[\hat{S}^{m|n}]$ is the zero of this addition. In fact, since $\hat{S}^{m|n} \setminus \hat{D}^{m|n} \cup_{\hat{S}^{m-1|n-1}} (\hat{S}^{m-1|n-1} \times \hat{D}^{1|1}) \cong \hat{D}^{m-1|n-1}$, we get

$$\begin{aligned} M \# \hat{S}^{m|n} &\cong \overline{M \setminus \hat{D}^{m|n}} \cup_{\hat{S}^{m-1|n-1}} (\overline{\hat{S}^{m|n} \setminus \hat{D}^{m|n}} \cup_{\hat{S}^{m-1|n-1}} (\hat{S}^{m-1|n-1} \times \hat{D}^{1|1})) \\ &\cong \overline{M \setminus \hat{D}^{m|n}} \cup_{\hat{S}^{m-1|n-1}} \hat{D}^{m|n} \cong M. \end{aligned}$$

Analogous calculus for M_C completes the proof. \square

In the following remark we will consider some examples and further results to better understand some relations between quantum homotopy superspheres and their classic limits.

Example 4.8 (Quantum homotopy $(7|n)$ -supersphere). *Let us calculate the extension classes for quantum homotopy $(7|n)$ -superspheres in the category $\mathfrak{Q}_{hyper,S}$, assumed classic regular, i.e., having the fiber bundle structure $\pi_C : \hat{\Sigma}^{7|n} \rightarrow \Sigma^7$.*

$$(39) \quad 0 \longrightarrow \hat{\Upsilon}_{7|n} \xrightarrow{\quad} \hat{\Theta}_{7|n} \xrightarrow{j_C} \Theta_7 \longrightarrow 0, \quad n \geq 0.$$

These are given by $H^2(\Theta_7; \hat{\Upsilon}_{7|n}) \cong H^2(\mathbb{Z}_{28}; \hat{\Upsilon}_{7|n})$. We get

$$(40) \quad H^2(\mathbb{Z}_{28}; \hat{\Upsilon}_{7|n}) = Hom_{\mathbb{Z}}(H_2(\mathbb{Z}_{28}; \mathbb{Z}); \hat{\Upsilon}_{7|n}) = Hom_{\mathbb{Z}}(0; \hat{\Upsilon}_{7|n}) \bigoplus Ext_{\mathbb{Z}}(H_1(\mathbb{Z}_{28}; \mathbb{Z}); \hat{\Upsilon}_{7|n}).$$

We shall prove that $Ext_{\mathbb{Z}}(H_1(\mathbb{Z}_{28}; \mathbb{Z}); \hat{\Upsilon}_{7|n}) = \hat{\Upsilon}_{7|n}/28 \cdot \hat{\Upsilon}_{7|n}$. (We have used the fact that $H_2(\mathbb{Z}_{28}; \mathbb{Z}) = 0$.) Let us look in some detail to this \mathbb{Z} -module. By using the projective resolution of \mathbb{Z}_{28} given in (41),

$$(41) \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\mu \cdot 28} \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}_{28} \longrightarrow 0$$

we get the exact sequence (42).

$$(42) \quad \begin{array}{ccccc} 0 & \longrightarrow & Hom_{\mathbb{Z}}(\mathbb{Z}_{28}; \hat{\Upsilon}_{7|n}) & \xrightarrow{\epsilon_*} & Hom_{\mathbb{Z}}(\mathbb{Z}; \hat{\Upsilon}_{7|n}) & \xrightarrow{\mu_*} & Hom_{\mathbb{Z}}(\mathbb{Z}; \hat{\Upsilon}_{7|n}) \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & Hom_{\mathbb{Z}}(\mathbb{Z}_{28}; \hat{\Upsilon}_{7|n}) & \xrightarrow{\epsilon_*} & \hat{\Upsilon}_{7|n} & \xrightarrow{\mu_*} & \hat{\Upsilon}_{7|n} \end{array}$$

Therefore we get

$$Ext_{\mathbb{Z}}(\mathbb{Z}_{28}; \hat{\Upsilon}_{7|n}) = \hat{\Upsilon}_{7|n}/\text{im}(\mu_*).$$

In order to see what is $\text{im}(\mu_*)$ we can use analogous considerations made in Example 5.6 in [83]. We get $\text{im}(\mu_*) = 28 \cdot \hat{\Upsilon}_{7|n}$, hence $Ext_{\mathbb{Z}}(\mathbb{Z}_{28}; \hat{\Upsilon}_{7|n}) = \hat{\Upsilon}_{7|n}/28 \cdot \hat{\Upsilon}_{7|n}$. The particular structure of this module, depends on the particular hypercomplex quantum superalgebra A considered. For example, take $A = \mathbb{C}$. One has $\hat{S}^{7|n} = \hat{S}^7$, hence, since $\hat{S}^7 \rightarrow S^7$, is just the fiber bundle $S^{14} \rightarrow S^7$, we can easily copy the result in Example 5.6 in [83], to conclude that $\hat{\Theta}_{7|n} = \hat{\Theta}_7 = \mathbb{Z}_2 \oplus \mathbb{Z}_{28}$, $\forall n \geq 0$.

Example 4.9 (Quantum homotopy $(m|n)$ -superspheres for the limit case $A = \mathbb{R}$). In the limit case where the quantum algebra is $A = \mathbb{R}$, for a quantum homotopy $(m|n)$ -supersphere $\hat{\Sigma}^{m|n}$ one has just $\hat{\Sigma}^{m|n} = \hat{\Sigma}_C^{m|n} = \Sigma^m$, hence $\pi_C = id_{\Sigma^m}$. Furthermore $\hat{\Theta}_{m|n} = \Theta_m$ and $\hat{\Upsilon}_{m|n} = 0 = [S^m] \in \Theta_m$. In particular if $m =$

$\{1, 2, 3, 4, 5, 6\}$, we get $\hat{\Theta}_{m|n} = \Theta_m = \hat{\Upsilon}_{m|n} = 0$, $\forall n \geq 0$. (For the smooth case $m = 4$ see [82].)

Theorem 4.10 (Homotopy groups of quantum $(m|n)$ -supersphere). *Quantum homotopy $(m|n)$ -superspheres cannot have, in general, the same homotopy groups of m -spheres:¹³*

$$(43) \quad \pi_k(\hat{\Sigma}^{m|n}) \cong \pi_k(\hat{S}^{m|n}) \neq \pi_k(S^m).$$

Furthermore, S^m can be identified with a contractible subspace, yet denoted S^m , of $\hat{S}^{m|n}$. There exists a mapping $\hat{S}^{m|n} \rightarrow S^m$, but this is not a retraction, and the inclusion $S^m \hookrightarrow \hat{S}^{m|n}$, cannot be a homotopy equivalence.

Proof. Since must necessarily be $\pi_k(\hat{\Sigma}^{m|n}) \cong \pi_k(\hat{S}^{m|n})$, $k \geq 0$, it is enough prove theorem for $\hat{S}^{m|n}$. We shall first recall some useful definitions and results of Algebraic Topology, here codified as lemmas.

Definition 4.11. *A pair (X, A) has the homotopy extension property if a homotopy $f_t : A \rightarrow Y$, $t \in I$, can be extended to homotopy $f_t : X \rightarrow Y$ such that $f_0 : X \rightarrow Y$ is a given map.*

Lemma 4.12. *If (X, A) is a CW pair, then it has the homotopy extension property.*

Lemma 4.13. *If the pair (X, A) satisfies the homotopy extension property and A is contractible, then the quotient map $q : X \rightarrow X/A$ is a homotopy equivalence.*

Let us consider that we can represent S^m into $\hat{S}^{m|n}$ by a continuous mapping $s : S^m \rightarrow \hat{S}^{m|n}$, defined by means of the commutative diagram in (44).

$$(44) \quad \begin{array}{ccc} \hat{S}^{m|n} & \xlongequal{\quad} & A^{m|n} \cup \{\infty\} \\ \pi_C \downarrow & & \uparrow s \equiv (\epsilon^m, 0, id_\infty) \\ S^m & \xlongequal{\quad} & \mathbb{R}^m \cup \{\infty\} \end{array}$$

where $\epsilon^m : \mathbb{R}^m \rightarrow A_0^m \subset A^m$ is induced by the canonical ring homomorphism $\epsilon : \mathbb{R} \rightarrow A$. s is a section of π : $\pi \circ s = id_{S^m}$. Let us yet denote by S^m the image of s . So we can consider the canonical couple $(\hat{S}^{m|n}, S^m)$ as a CW pair, hence it has the homotopy extension property. S^m is not a contractible subcomplex of $\hat{S}^{m|n}$, so in general the quotient map $\hat{q} : \hat{S}^{m|n} \rightarrow \hat{S}^{m|n}/S^m$ is not a homotopy equivalence. We have the following lemma.

Lemma 4.14. *The couple (S^m, ∞) can be deformed into $(\hat{S}^{m|n}, \infty)$ to the base point $\{\infty\}$.*

Proof. In fact, let $p \in \hat{S}^{m|n} \setminus S^m$. Then the inclusion $i : S^m \hookrightarrow \hat{S}^{m|n}$ is nullhomotopic since $\hat{S}^{m|n} \setminus \{\infty\} \approx A^{m|n}$ (homeomorphism). \square

Since S^m is contractible into $\hat{S}^{m|n}$, to the point $\infty \in \hat{S}^{m|n}$, the quotient map $\hat{q} : \hat{S}^{m|n} \rightarrow \hat{S}^{m|n}/S^m$ can be deformed into quotient mapping \hat{q}_t over deformed quotient spaces $X_t \equiv \hat{S}^{m|n}/S_t^m$, with $S_t^m \equiv f_t(S^m) \subset \hat{S}^{m|n}$, for some homotopy $f : I \times S^m \rightarrow \hat{S}^{m|n}$, such that $X_0 = \hat{S}^{m|n}/S^m$, $X_1 = \hat{S}^{m|n}$ and $\hat{q}_1 = id_{\hat{S}^{m|n}}$. (See diagram (45).)

¹³In other words, quantum homotopy $(m|n)$ -superspheres are not homotopy equivalent to the m -sphere.

$$(45) \quad \begin{array}{ccc} \hat{S}^{m|n} & \xrightarrow{\hat{q}_0=\hat{q}} & \hat{S}^{m|n}/S^m \equiv X_0 \\ & \searrow \hat{q}_t & \vdots \\ & & \hat{S}^{m|n}/S_t^m \equiv X_t \\ & \searrow \hat{q}_1 & \vdots \\ & & \hat{S}^{m|n}/\{\infty\} = \hat{S}^{m|n} \equiv X_1 \end{array}$$

But this does not assure that \hat{q} is a homotopy equivalence.¹⁴ Let us, now, consider also some further lemmas.

Lemma 4.15. *If (X, A) is a CW pair and we have attaching maps $f, g : A \rightarrow X_0$ that are homotopic, then $X_0 \cup_f X_1 \simeq X_0 \cup_g X_1 \text{ rel } X_0$ (homotopy equivalence).*

Lemma 4.16. *If (X, A) satisfies the homotopy extension property and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X .*

Lemma 4.17. *A map $f : X \rightarrow Y$ is a homotopy equivalence iff X is a deformation retract of the mapping cylinder M_f .*

Let us emphasize that we have a natural continuous mapping $\pi_C : \hat{S}^{m|n} \rightarrow S^m$, i.e., the surjection between the quantum $(m|n)$ -supersphere and its classic limit, identified by the commutative diagram (44). The inclusion $i : S^m \hookrightarrow \hat{S}^{m|n}$ cannot be a deformation retract (and neither a strong deformation retract), otherwise i should be a homotopy equivalence.¹⁵ However, $\pi_C : \hat{S}^{m|n} \rightarrow S^m$, cannot be neither a retraction, otherwise their homotopy groups should be related by the split short exact sequence (46),

$$(46) \quad 0 \longrightarrow \pi_k(S^m, \infty) \xrightleftharpoons[r_*]{i_*} \pi_k(\hat{S}^{m|n}, \infty) \longrightarrow \pi_k(\hat{S}^{m|n}, S^m, \infty) \longrightarrow \infty$$

hence we should have the splitting given in (47). (For details on relations between homotopy groups and retractions see, e.g. [67].)

$$(47) \quad \pi_k(\hat{S}^{m|n}, \infty) \cong \text{im}(i_*) \bigoplus \ker(r_*) \cong \pi_k(S^m, \infty) \bigoplus \ker(r_*).$$

But this cannot work. In fact, in the case $A = \mathbb{C}$, we should have the commutative diagram (48) with exact horizontal lines.

$$(48) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(S^1, \infty) & \xrightleftharpoons[r_*]{i_*} & \pi_1(\hat{S}^1, \infty) & \longrightarrow & \pi_1(\hat{S}^1, S^1, \infty) \longrightarrow \infty \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \pi_1(\hat{S}^1, S^1, \infty) \longrightarrow 0 \end{array}$$

¹⁴Rally S^m is contractible in $\hat{S}^{m|n}$, but is not a contractible sub-complex of $\hat{S}^{m|n}$. This clarifies the meaning of Lemma 4.13. For example, in the case $A = \mathbb{C}$, one has that $\hat{S}^{1|n}/S^1$ is not homotopy equivalent to $S^2 \cong_A \hat{S}^{1|n}$. In fact $\pi_2(S^2) = \mathbb{Z}$ and $\pi_2(S^2/S^1) \cong \pi_2(S^2 \vee S^2) \cong H_2(S^2 \vee S^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

¹⁵It is enough to consider the counterexample when $A = \mathbb{C}$ and $\hat{S}^{1|n} = \mathbb{C} \cup \{\infty\} = \mathbb{R}^2 \cup \{\infty\} = S^2$. Then S^1 cannot be homotopy equivalent to $\hat{S}^{1|n} = S^2$, since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(S^2) = 0$.

This should imply that $\pi_1(S^1, \infty) = 0$, instead that \mathbb{Z} , hence the bottom horizontal line in (48) cannot be an exact sequence, hence $\pi_C : \hat{S}^1 \cong S^2 \rightarrow S^1$ cannot be a retraction ! \square

Corollary 4.18. *Quantum homotopy superspheres cannot be homotopy equivalent to S^m , except in the case that the quantum algebra A reduces to \mathbb{R} .*

Quantum homotopy groups for quantum supermanifolds are introduced in [75].

Theorem 4.19 (Quantum homotopy groups of quantum $(m|n)$ -supersphere). *Quantum homotopy $(m|n)$ -superspheres have quantum homotopy groups isomorphic to homotopy groups of m -spheres:*

$$(49) \quad \hat{\pi}_k(\hat{S}^{m|n}) = \hat{\pi}_k(\hat{S}^{m|n}) \cong \pi_k(S^m).$$

Proof. In fact, we can prove for examples that $\hat{\pi}_k(\hat{S}^{m|n}) \cong 0$, for $k < m$, and $\hat{\pi}_m(\hat{S}^{m|n}) \cong \mathbb{Z}$. For this it is enough to reproduce the analogous proofs for the commutative spheres, by substituting cells with quantum supercells. For example we can have the following quantum versions of analogous propositions for commutative CW complexes.

Lemma 4.20. *Let X be a quantum CW-complex admitting a decomposition in two quantum subcomplexes $X = A \cup B$, such that $A \cap B = C \neq \emptyset$. If (A, C) is m -connected and (B, C) is n -connected, $m, n \geq 0$, then the mappings $\hat{\pi}_k(A, C) \rightarrow \hat{\pi}_k(X, B)$ induced by inclusion is an isomorphism for $k < m + n$, and a surjection for $k = m + n$.*

Lemma 4.21 (Quantum Freudenthal suspension theorem). *The quantum suspension map $\hat{\pi}_k(\hat{S}^{m|n}) \rightarrow \hat{\pi}_{k+1}(\hat{S}^{m+1|n+1})$ is an isomorphism for $k < 2m - 1$, and a surjection for $k = 2m - 1$.¹⁶*

As a by-product we get the isomorphism $\hat{\pi}_m(\hat{S}^{m|n}) \cong \mathbb{Z}$. \square

Remark 4.22. *Let us emphasize that Theorem 4.19 does not allow to state that S^m is a deformation retract of $\hat{S}^{m|n}$, as one could conclude by a wrong application of the Whitehead's theorem, reported in the following lemma.*

Lemma 4.23 (Whitehead's theorem). *If a map $f : X \rightarrow Y$ between connected CW complexes induces isomorphisms $f_* : \pi_m(X) \rightarrow \pi_m(Y)$ for all m , then f is a homotopy equivalence. Furthermore, if f is the inclusion of a subcomplex $f : X \hookrightarrow Y$, then X is a deformation retract of Y .*

In fact, in the case $i : S^m \hookrightarrow \hat{S}^{m|n}$ we are talking about different CW structures. One for S^m is the usual one, the other, for $\hat{S}^{m|n}$ is the quantum super-CW structure. In order to easily understand the difference let us refer again to the case $A = \mathbb{C}$. Here one has $\pi_1(S^1) = \mathbb{Z} = \hat{\pi}_1(\hat{S}^{1|0} = \hat{S}^1)$, but $\hat{\pi}_1(\hat{S}^1) = [\hat{S}^1, \hat{S}^1] = [S^2, S^2] = \pi_2(S^2)$. Furthermore, $\pi_1(\hat{S}^1) = [S^1, S^2] = 0 \neq \pi_1(S^1)$. Therefore, $\hat{S}^{1|n}$, with respect to the usual CW complex structure, has its first homotopy group zero, hence different from the first homotopy group of its classic limit S^1 . In fact S^1 is not a deformation retract of $S^2 = \hat{S}^1$. (Therefore there is not contradiction with the Whitehead's theorem.)

¹⁶This holds also for quantum suspension $\hat{\pi}_k(X) \rightarrow \hat{\pi}_{k+1}(\hat{S}X)$, for an $(m-1)$ -connected quantum CW-complex X .

Moreover, it is useful to formulate the quantum version of the Whitehead's theorem and some related lemmas. These can be proved by reproducing analogous proofs by substituting CW complex structure with quantum CW complex structure in quantum supermanifolds.

Theorem 4.24 (Quantum Whitehead theorem). *If a map $f : X \rightarrow Y$ between connected quantum CW complexes induces isomorphisms $f_* : \hat{\pi}_m(X) \rightarrow \hat{\pi}_m(Y)$ for all m , then f is a homotopy equivalence. Furthermore, if f is the inclusion of a quantum subcomplex $f : X \hookrightarrow Y$, then X is a quantum deformation retract of Y .*

Lemma 4.25 (Quantum compression lemma). *Let (X, A) be a quantum CW pair and let (Y, B) be any quantum pair with $B \neq \emptyset$. Let us assume that for each m $\hat{\pi}_m(Y, B, y_0) = 0$, for all $y_0 \in B$, and $X \setminus A$ has quantum supercells of dimension $(m|n)$. Then, every map $f : (X, A) \rightarrow (Y, B)$ is homotopic rel_A to a map $X \rightarrow B$.¹⁷*

Lemma 4.26 (Quantum extension lemma). *Let (X, A) be a quantum CW pair and let $f : A \rightarrow Y$ be a mapping with Y a path-connected quantum supermanifold. Let us assume that $\hat{\pi}_{m-1}(Y) = 0$, for all m , such that $X \setminus A$ has quantum cells of dimension m . Then, f can be extended to a map $f : X \rightarrow Y$.*

Proof. The proof can be done inductively. Let us assume that f has been extended over the quantum $(m-1|n-1)$ -superskeleton. Then, an extension over quantum $(m|n)$ -supercells exists iff the composition of the quantum supercell's attaching map $\hat{S}^{m-1|n-1} \rightarrow \hat{X}^{m-1|n-1}$ with $f : \hat{X}^{m-1|n-1} \rightarrow Y$ is null homotopic. \square

As a by-product of above results we get also the following theorems that relate quantum homotopy groups and quantum relative homotopy groups.

Theorem 4.27 (Quantum exact long homotopy sequence). *One has the exact sequence (50).*

$$(50) \quad \begin{array}{ccccccc} \cdots & & \hat{\pi}_m(A, x_0) & \xrightarrow{\hat{i}_*} & \hat{\pi}_m(X, x_0) & \xrightarrow{\hat{j}_*} & \hat{\pi}_m(X, A, x_0) \\ & & & & & & \downarrow \hat{\partial} \\ & & \hat{\pi}_0(X, x_0) & \longleftarrow \cdots \longleftarrow & \hat{\pi}_{n-1}(A, x_0) & & \end{array}$$

where \hat{i}_* and \hat{j}_* are induced by the inclusions $\hat{i} : (A, x_0) \hookrightarrow (X, x_0)$ and $\hat{j} : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ respectively. Furthermore, $\hat{\partial}$ comes from the following composition $(\hat{S}^{m-1|n-1}, s_0) \hookrightarrow (\hat{D}^{m|n}, \hat{S}^{m-1|n-1}, s_0) \rightarrow (X, A, x_0)$, hence $\hat{\partial}[f] = [f|_{\hat{S}^{m-1|n-1}}]$.

Theorem 4.28 (Quantum Hurewicz theorem). *The exact commutative diagram in (51) relates (quantum) homotopy groups and (quantum) homology groups for (quantum) homotopy $(m|n)$ -spheres, $m \geq 2$. The morphisms a and b are isomorphisms*

¹⁷When $m = 0$, the condition $\hat{\pi}_m(Y, B, y_0) = 0$, for all $y_0 \in B$, means that (Y, B) is 0-connected. Let us emphasize that there is not difference between 0-connected and quantum 0-connected. In fact $[\hat{S}^{0|n}, Y] = \hat{\pi}_0(Y) = \pi_0(Y) = [S^0, Y]$, since $\hat{S}^{0|n} \simeq S^0 = (\{a\}, \{b\})$, i.e., homotopy equivalent to a set of two points. However, after Theorem 4.19, there is not difference between the notion of *quantum p -connected* (i.e., $\hat{\pi}_k = 0$, $k \leq p$), quantum (homotopy) $(m|n)$ -supersphere, and *p -connected* (i.e., $\pi_k = 0$, $k \leq p$), (homotopy) $(m|n)$ -supersphere. In other words, a quantum homotopy $(m|n)$ -supersphere is quantum $(m-1)$ -connected as well as its classic limit is $(m-1)$ -connected.

for $p \leq m$ and epimorphisms for $p = m + 1$.¹⁸

$$(51) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \pi_p(S^m) & \xrightleftharpoons{\quad} & \hat{\pi}_p(\hat{S}^{m|n}) & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \\ 0 & \longrightarrow & H_p(S^m; \mathbb{Z}) & \xrightleftharpoons{\quad} & H_p(\hat{S}^m; \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

The following propositions are also stated as direct results coming from Theorem 4.19 and analogous propositions for topologic spaces.

Proposition 4.29. *The following propositions are equivalent for $i \leq m - 1$.*

- 1) $S^i \rightarrow S^n$ is homotopic to a constant map.
- 2) $S^i \rightarrow S^n$ extends to a map $D^{i+1} \rightarrow S^n$.
- 3) $\hat{S}^{i|j} \rightarrow \hat{S}^{m|n}$ is homotopic to a constant map.
- 4) $\hat{S}^{i|j} \rightarrow \hat{S}^{m|n}$ extends to a map $\hat{D}^{i+1|j+1} \rightarrow \hat{S}^{m|n}$.

Proof. 1) and 2) follow from the fact that S^m is $(m - 1)$ -connected, and 3) and 4) from the fact that $\hat{S}^{m|n}$ is quantum $(m - 1)$ -connected. Furthermore, let us recall the following related result of Algebraic Topology.¹⁹

Lemma 4.30. *The following propositions are equivalent.*

- (i) The space X is m -connected.
- (ii) Every map $f : S^i \rightarrow X$ is homotopic to a constant map.
- (ii) Every map $f : S^i \rightarrow X$ extends to a map $D^{i+1} \rightarrow X$.

□

Let us, now, consider quantum super PDE's, in the category $\mathfrak{Q}_{hyper,S}$, with respect to quantum homotopy $(m|n)$ -superspheres.

Definition 4.31 (Quantum hypercomplex exotic super PDE's). *Let $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ be a k -order PDE on the fiber bundle $\pi : W \rightarrow M$ in the category $\mathfrak{Q}_{hyper,S}$, with $\dim_A M = (m|n)$ and $\dim_B W = (n|m, r|s)$, where $B = A \times E$ and E is also a $Z(A)$ -module. We say that \hat{E}_k is a quantum exotic PDE if it admits Cauchy integral manifolds $N \subset \hat{E}_k$, $\dim N = (m - 1|n - 1)$, such that one of the following two conditions is verified.*

- (i) $\hat{\Sigma}^{m-2|n-2} \equiv \partial N$ is a quantum exotic supersphere of dimension $(m - 2|n - 2)$, i.e. $\hat{\Sigma}^{m-2|n-2}$ is homeomorphic to $\hat{S}^{m-2|n-2}$, $(\Sigma^{m-2|n-2} \approx \hat{S}^{m-2|n-2})$ but not diffeomorphic to $\hat{S}^{m-2|n-2}$, $(\hat{\Sigma}^{m-2|n-2} \not\approx \hat{S}^{m-2|n-2})$.
- (ii) $\emptyset = \partial N$ and $N \approx \hat{S}^{m-1|n-1}$, but $N \not\approx \hat{S}^{m-1|n-1}$.²⁰

¹⁸Compare with analogous theorem in [75] for quantum supermanifolds.

¹⁹There exists also a relative version of Lemma 4.30, saying that $\pi_i(X, A, x_0) = 0$, for all $x_0 \in A$, is equivalent to one of the following propositions. (a1) Every map $(D^i, \partial D^i) \rightarrow (X, A)$ is homotopic rel ∂D^i , to a map $D^i \rightarrow A$. (a2) Every map $(D^i, \partial D^i) \rightarrow (X, A)$ is homotopic through such maps to a map $D^i \rightarrow A$. (a3) Every map $(D^i, \partial D^i) \rightarrow (X, A)$ is homotopic through such maps to a constant map $D^i \rightarrow A$.

²⁰For complementary information see [83].

Definition 4.32 (Quantum hypercomplex exotic-classic super PDE's). *Let $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ be a k -order super PDE as in Definition 4.31. We say that \hat{E}_k is a quantum exotic-classic super PDE if it is a quantum exotic super PDE, and the classic limit of the corresponding Cauchy quantum exotic supermanifolds are also exotic homotopy spheres.*

From above results we get also the following one.

Lemma 4.33. *A quantum super PDE $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$, where m is such that $\Theta_{m-1} = 0$, cannot be a quantum exotic-classic PDE, in the sense of Definition 4.32.*

Lemma 4.34. *For $m \in \{1, 2, 3, 4, 5, 6\}$, one has the isomorphism reported in (52).*

$$(52) \quad \hat{\Theta}_{m|n} \cong \hat{\Upsilon}_{m|n}.$$

In correspondence of such dimensions on m we cannot have quantum exotic-classic super PDE's.

Proof. Isomorphisms in (52), follow directly from above lemmas, and the fact that $\Theta_m = 0$ for $m \in \{1, 2, 3, 4, 5, 6\}$. (See Refs. [80, 82].) \square

Example 4.35 (The quantum hypercomplex Ricci flow super-equation). *As a by-product of Theorem 4.6 it follows that under the same hypotheses there adopted on the quantum algebra A , it follows that the quantum Ricci flow equation in the category $\mathfrak{Q}_{hyper,S}$, is a quantum exotic super PDE. On the other hand, such a PDE cannot be quantum exotic-classic for $m < 7$. (See [79, 80, 82].) (For complementary information on the Ricci flow equation see also the following Refs. [2, 18, 28, 29, 30, 31, 32, 36, 43, 44, 45, 47, 48, 49, 51, 52, 54, 92, 97, 98].)*

Example 4.36 (The quantum hypercomplex Navier-Stokes super-equation). *The quantum Navier-Stokes equation can be encoded on the quantum super-extension of the affine fiber bundle $\pi : W \equiv M \times \mathbf{I} \times \mathbb{R}^2 \rightarrow M$, $(x^\alpha, \dot{x}^i, p, \theta)_{0 \leq \alpha \leq 3, 1 \leq i \leq 3} \mapsto (x^\alpha)$. (See Refs. [61, 63, 82] for the Navier-Stokes equation in the category of commutative manifolds and [65, 67, 83] for its quantum extension on quantum manifolds.) Therefore, Cauchy manifolds are (3|3)-dimensional quantum supermanifolds. For such dimension do not exist exotic spheres. Therefore, the Navier-Stokes equation cannot be a quantum exotic-classic super PDE. Similar considerations hold for PDE's of the quantum super-extensions of continuum mechanics PDE's.*

Example 4.37 (The quantum hypercomplex $(m|n)$ -d'Alembert super-equation). *The quantum $(m|n)$ -d'Alembert super-equation on $A^{m|n}$ cannot be a quantum exotic-classic super PDE for quantum $(m|n)$ -dimensional Riemannian manifolds, with $m < 7$, in the category $\mathfrak{Q}_{hyper,S}$. (For complementary information on the geometric structure of the d'Alembert PDE in the category of commutative manifolds and quantum manifolds see Refs. [63, 64, 67, 69, 81, 82, 83].)*

Example 4.38 (The quantum hypercomplex Einstein super-equation). *Considerations similar to ones made in Example 4.37, hold for the quantum Einstein super-equation in the category $\mathfrak{Q}_{hyper,S}$.*

Theorem 4.39 (Integral bordism groups in quantum hypercomplex exotic super PDE's in the category $\mathfrak{Q}_{hyper,S}$ and stability). *Let $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ be a quantum*

exotic formally integrable and completely integrable super PDE on the fiber bundle $\pi : W \rightarrow M$, in the category $\mathfrak{Q}_{hyper,S}$, such that $\hat{g}_k \neq 0$ and $\hat{g}_{k+1} \neq 0$.²¹ Then there exists a bi-graded topologic spectrum $\Xi_{i|j}$ such that for the singular integral $(p|q)$ -(co)bordism groups can be expressed by means of suitable bigraded homotopy groups as reported in (53).

$$(53) \quad \left\{ \begin{array}{l} \Omega_{p|q,s}^{\hat{E}_k} = \lim_{(i|j) \rightarrow (\infty|\infty)} \hat{\pi}_{p+i|q+j}(\hat{E}_k^+ \wedge \Xi_{i|j}) \\ \Omega_{\hat{E}_k}^{p|q,s} = \lim_{(i|j) \rightarrow (\infty|\infty)} [\hat{S}^{i|j} \hat{E}_k^+, \Xi_{p+i|q+j}] \end{array} \right\}_{p \in \{0,1,\dots,m-1\}, q \in \{0,1,\dots,n-1\}}.$$

Furthermore, the singular integral bordism group for admissible smooth closed compact Cauchy manifolds, $N \subset \hat{E}_k$, is given in (54).

$$(54) \quad \Omega_{m-1|n-1,s}^{\hat{E}_k} \cong H_{m-1|n-1}(W; A).$$

In the quantum homotopy equivalence full admissibility hypothesis, i.e., by considering admissible only $(m-1|n-1)$ -dimensional smooth Cauchy integral supermanifolds identified with quantum homotopy superspheres, and assuming that the space of conservation laws is not trivial, one has $\Omega_{m-1|n-1,s}^{\hat{E}_k} = 0$. Then \hat{E}_k becomes a quantum extended 0-crystal super PDE. Therefore, there exists a global singular attractor, in the sense that all Cauchy supermanifolds, identified with quantum homotopy $(m-1|n-1)$ -superspheres, bound singular manifolds.

Furthermore, if in W we can embed all the quantum homotopy $(m-1|n-1)$ -superspheres, and all such supermanifolds identify admissible smooth $(m-1|n-1)$ -dimensional Cauchy supermanifolds of \hat{E}_k , then two of such Cauchy supermanifolds bound a smooth solution iff they are diffeomorphic and one has the following bijective mapping: $\Omega_{m-1|n-1}^{\hat{E}_k} \leftrightarrow \hat{\Theta}_{m-1|n-1}$.

Moreover, if in W we cannot embed all quantum homotopy $(m-1|n-1)$ -superspheres, but only $\hat{S}^{m-1|n-1}$, then in the quantum supersphere full admissible hypothesis, i.e., by considering admissible only quantum $(m-1|n-1)$ -dimensional smooth Cauchy integral supermanifolds identified with $\hat{S}^{m-1|n-1}$, then $\Omega_{m-1|n-1}^{\hat{E}_k} = 0$. Therefore \hat{E}_k becomes a quantum 0-crystal super PDE and there exists a global smooth attractor, in the sense that two of such smooth Cauchy supermanifolds, identified with $\hat{S}^{m-1|n-1}$ bound quantum smooth supermanifolds. Instead, two Cauchy supermanifolds identified with quantum exotic $(m-1|n-1)$ -superspheres bound by means of quantum singular solutions only.

All above quantum smooth or quantum singular solutions are unstable. Quantum smooth solutions can be stabilized.

Proof. The relations (53) and (54) can be proved by a direct extension of analogous characterizations of integral bordism groups of PDE's in the category of commutative manifolds and quantum PDEs. (See [63, 64, 82, 83].) Then the rest of the proof follows directly by using above results in this section, and following a road similar to the proof of Theorem 5.38 given in [83]. \square

Similarly one can prove the following theorem that extends in the category $\mathfrak{Q}_{hyper,S}$ an analogous theorem in the category of commutative manifolds and in the category \mathfrak{Q}_{hyper} . (See [82, 83].)

²¹The fiber bundle $\pi : W \rightarrow M$ is as in Definition 4.31, hence $\dim_A M = (m|n)$, $\dim_B W = (n|m, r|s)$, with E endowed with a $Z(A)$ -module structure too.

Theorem 4.40 (Integral h-cobordism in quantum hypercomplex Ricci flow super PDE's). *The quantum Ricci flow equation for quantum $(m|n)$ -dimensional Riemannian supermanifolds, admits that starting from a quantum $(m|n)$ -dimensional supersphere $\hat{S}^{m|n}$, we can dynamically arrive, into a finite time, to any quantum $(m|n)$ -dimensional homotopy supersphere M . When this is realized with a smooth solution, i.e., solution with characteristic flow without singular points, then $\hat{S}^{m|n} \cong M$. The other quantum homotopy spheres $\hat{\Sigma}^{m|n}$, that are homeomorphic to $\hat{S}^{m|n}$ only, are reached by means of singular solutions.*

For $1 \leq m \leq 6$, quantum hypercomplex Ricci flow super PDE's cannot be quantum exotic-classic ones. In particular, the case $m = 4$, is related to the proof that the smooth Poincaré conjecture is true.

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STRONG REACTIONS IN QUANTUM SUPER PDE'S.II: NONLINEAR QUANTUM PROPAGATORS

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ABSTRACT. In this second part, of a work in three parts, devoted to encode strong reactions of the high energy physics, in the algebraic topologic theory of quantum super PDE's, (previously formulated by A. Prástaro), decomposition theorems of integral bordisms in quantum super PDEs are obtained. (For part I and part III see [88, 89].) In particular such theorems allow us to obtain representations of quantum nonlinear propagators in quantum super PDE's, by means of elementary ones (*quantum handle decompositions* of quantum nonlinear propagators). These are useful to encode nuclear and subnuclear reactions in quantum physics. Prástaro's geometric theory of quantum PDE's allows us to obtain constructive and dynamically justified answers to some important open problems in high energy physics. In fact a *Regge-type* relation between *reduced quantum mass* and *quantum phenomenological spin* is obtained. A dynamical *quantum Gell-Mann-Nishijima formula* is given. An existence theorem of observed local and global solutions with electric-charge-gap, is obtained for quantum super Yang-Mills PDE's, $(YM)[i]$, by identifying a suitable constraint, $(YM)[i]_w \subset (YM)[i]$, *quantum electromagnetic-Higgs PDE*, bounded by a quantum super partial differential relation $(Goldstone)[i]_w \subset (YM)[i]$, *quantum electromagnetic Goldstone-boundary*. An electric neutral, connected, simply connected observed quantum particle, identified with a Cauchy data of $(YM)[i]$, it is proved do not belong to $(YM)[i]_w$. Existence of *Q-exotic quantum nonlinear propagators* of $(YM)[i]$, i.e., quantum nonlinear propagators that do not respect the quantum electric-charge conservation is obtained.

By using integral bordism groups of quantum super PDE's, a *quantum crossing symmetry theorem* is proved. As a by-product existence of *massive photons* and *massive neutrinos* are obtained. A dynamical proof that quarks can be broken-down is given too. A *quantum time*, related to the observation of any quantum nonlinear propagator, is calculated. Then an *apparent quantum time* estimate for any reaction is recognized. A criterion to identify solutions of the quantum super Yang-Mills PDE encoding *(de)confined quantum systems* is given. *Supersymmetric particles* and *supersymmetric reactions* are classified on the ground of integral bordism groups of the quantum super Yang-Mills PDE (YM) . Finally, existence of the *quantum Majorana neutrino* is proved. As a by-product, the existence of a new quasi-particle, that we call *quantum Majorana neutralino*, is recognized made by means of two quantum Majorana neutrinos, a couple $(\tilde{\nu}_e, \tilde{\bar{\nu}}_e)$, supersymmetric partner of $(\nu_e, \bar{\nu}_e)$, and two Higgsinos.

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1. Introduction

The algebraic topologic theory of quantum (super) PDE's formulated by A. Prástaro, allows to directly encode quantum phenomena in a category of noncommutative manifolds (quantum (super)manifolds) and to finally solve the problem of unification, at quantum level, of gravity with the fundamental forces [65, 68, 69, 70, 71, 72, 74, 77, 79, 81, 82]. In particular, this theory allowed to recognize the mechanism of mass creation/destruction, as a natural geometric phenomenon related to the algebraic topologic structure of quantum (super) PDEs encoding the quantum system under study [82].

Aim of this second part is to explicitly prove that nuclear and subnuclear reactions can be encoded as boundary value problems in the Prástaro's algebraic topology of quantum super PDEs, and can be represented in terms of elementary reactions. (For part I and part III see [88, 89].)

It is important to emphasize that quantum conservation laws do not necessarily produce conservation of quantum charges in quantum reactions. In fact, it is also important consider the topological structure of the corresponding quantum nonlinear propagators encoding these reactions. In [88] (Part I), we have shown this fact for the observed quantum energy. In this second part we characterize observed quantum nonlinear propagators V of the observed quantum super Yang-Mills PDE, $(\widehat{YM})[i]$, with respect to the total quantum electric-charge. Then we define Q -exotic quantum nonlinear propagators ones where there is a non-zero lost quantum electric-charge, $\mathfrak{Q}[V] \in A$, in the corresponding encoded reactions. (A is the fundamental quantum superalgebra in $(\widehat{YM})[i]$.) This important phenomenon, that is related to the gauge invariance of $(\widehat{YM})[i]$, was non-well previously understood, since the gauge invariance was wrongly interpreted as a condition that necessarily produce the conservation of electric-charge in reactions. Really just the gauge invariance is the main origin of such phenomenon, but beside the structure of the quantum nonlinear propagator. This fundamental aspect of quantum reactions in $(\widehat{YM})[i]$, gives strong theoretical support to the guess about existence of quantum reactions where the "electric-charge" is not conserved. This was quasi a dogma in particle physics. However, there are in the world many heretical experimental efforts to prove existence of decays like the following $e^- \rightarrow \gamma + \nu$, i.e. electron decay into a photon and neutrino. In this direction some first weak experimental evidences were recently obtained.¹ With this respect, one cannot remark the singular role played, in the history of the science in these last 120 years, by the electron, a very

¹See [25]. Some other exotic decays were also investigated, as for example the exotic neutron's decay: $n \rightarrow p + \nu + \bar{\nu}$. [54].

small and light particle. In fact, at the beginning of the last century was just the electron to cause break-down in the Maxwell and Lorentz physical picture of the micro-world, until to produce a completely new point of view, i.e. the quantum physics. Now, after 120 years the electron appears to continue do not accept the place that physicists have reserved to it in the *world-puzzle*.

In the following we show as it is organized the paper and list the main results.

2. Theorem 2.2: A representation of quantum nonlinear propagators by means *quantum exchangions* and *quantum virtual particles* is given. 3. Theorem 3.2 characterizes quantum nonlinear propagators in quantum super Yang-Mills PDE, on quantum super Minkowski space-time. 4. Theorem 4.5: The quantum mass is represented by means of the quantum torsion of the corresponding solution. Corollary 4.7: A direct relation between square *reduced quantum mass* and *phenomenological quantum spin* is given.² Theorem 4.8 gives a dynamical *quantum Gell-Mann-Nishijima formula*. By means of this formula one obtains a dynamical interpretation of *quantum hypercharge* and *quantum 3th isospin component*. An existence theorem of observed local and global solutions with electric-charge-gap, is obtained too for observed quantum super Yang-Mills PDE's, $(\widehat{YM})[i]$. It is identified a suitable constraint, $(\widehat{YM})[i]_w \subset (\widehat{YM})[i]$, *quantum electromagnetic-Higgs PDE*, bounded by a quantum super partial differential relation $(\widehat{Goldstone})_w \subset (\widehat{YM})[i]$, *quantum electromagnetic Goldstone-boundary*. An observed quantum nonlinear propagator $V \subset (\widehat{YM})[i]$, crossing the quantum electromagnetic Goldstone-boundary loses (or acquires) the property to have an electric-charge gap. An electric neutral, connected, simply connected observed quantum particle, identified with a Cauchy data of $(\widehat{YM})[i]$, cannot be contained into $(\widehat{YM})[i]_w$. Theorem 4.10 proves that the quantum electric charge is not necessarily a conserved quantum law for any quantum nonlinear propagator of $(\widehat{YM})[i]$. In fact, it is proved the existence of *Q-exotic quantum nonlinear propagators* in $(\widehat{YM})[i]$, encoding observed reactions that do not respect the conservation of the quantum electric-charge. Theorem 4.12: By using integral bordism groups of quantum super PDE's, a *quantum crossing symmetry theorem* is proved. Theorem 4.13: Existence of solutions of quantum super Yang-Mills PDEs, (\widehat{YM}) , representing productions of *quantum massive photons*, is proved.³ Theorem 4.15: Theorem 4.13 is generalized to other quantum electric-charged particles. Theorem 4.18 states existence of quantum massive neutrinos, i.e., there exist quantum nonlinear propagators of (\widehat{YM}) encoding decays of massive quasi-particles into a couple (neutrino, antineutrino).⁴ Theorem 4.20: It is proved that quarks cannot be considered fundamental particles, i.e., they can be broken-down. Theorem 4.23: Existence of massive quasi-particles, with masses

²In particular this result shows how the phenomenological *Regge-type trajectories* emerge from the geometric theory of quantum (super) PDE's formulated by A. Prástaro.

³These can be identified with quantum neutral massive vector bosons. The annihilation electron-positron must necessarily produce an intermediate *quantum virtual massive photon*.

⁴Quantum massive photons, quantum massive neutrinos and *a*-quantum massive photons could interpret the so-called "dark matter" that nowadays is a spellbinding object of active research. This exotic matter is related to the geometric structure of (\widehat{YM}) that identifies the subequation (\widehat{Higgs}) . A global solution $V \subset (\widehat{YM})$, crossing the quantum Goldstone-boundary of (\widehat{Higgs}) , acquires (or loses) mass. (See [82], Example 4.19 and footnote at page 30, concerning π^+ -photoproduction.)

that apparently contradict the conservation of mass-energy, is proved. This is related to the new concept of *quantum time* (see Definition 4.21) that comes from the interaction between quantum relativistic frame (i.e., an observer) and the quantum system. Theorem 4.27: A criterion to identify solutions of (\widehat{YM}) , encoding confined quantum systems, is proved. Lemma 4.31 and Lemma 4.32 state existence of supersymmetric particles and supersymmetric reactions by means of integral bordism groups of (\widehat{YM}) . Theorem 4.29: Existence of *quantum Majorana neutrino* and a new quasi-particle, that we call *quantum Majorana neutralino*, for its similarity with the so-called neutralino, are obtained, by using algebraic topologic properties of (\widehat{YM}) .

2. Surgery in Quantum Super PDEs

Let us assume as a prerequisite of this section the knowledge of some previous works by A. Prástaro on quantum (super) PDEs. However, to fix more directly ideas we recall also some fundamental definitions and results that are soon related to the subject here considered.

Let $A = A_0 \times A_1$ be a quantum superalgebra in the sense of Prástaro.

Definition 2.1. *An $(m|n)$ -dimensional bordism in the category \mathfrak{Q}_S of quantum supermanifolds, consists of the following $(W; M_0, f_0; M_1, f_1)$, where W is a compact quantum supermanifold of dimension $(m|n)$ and closed $(m-1|n-1)$ -dimensional quantum supermanifolds M_0 and M_1 , such that $\partial W = N_0 \sqcup N_1$, and quantum diffeomorphisms $f_i : M_i \rightarrow N_i$, $i = 0, 1$. An $(m|n)$ -dimensional h -bordism, (resp. s -bordism) is a $(m|n)$ -dimensional bordism as above, such that the inclusions $N_i \hookrightarrow W$, $i = 0, 1$, are homotopy equivalences, (resp. simply homotopy equivalences). We will simply denote also by $(W; M_0, M_1)$ an $(m|n)$ -dimensional bordism in the category \mathfrak{Q}_S .*

Theorem 2.2 (Quantum handle decomposition of nonlinear quantum propagator in quantum super PDEs). *Let $\hat{E}_k \subset \hat{J}_{m+1|n+1}^k(W)$ be a quantum super PDE in the category \mathfrak{Q}_S and let $V \subset \hat{E}_k$ be a compact solution of a boundary value problem, i.e., $\partial V = N_0 \sqcup N_1$.⁵ Let us assume that \hat{E}_k is formally integrable and completely integrable. Then there exists a quantum-super-handle-presentation (1) of the nonlinear quantum propagator V .*

$$(1) \quad V_1 \bigcup V_2 \bigcup \cdots \bigcup V_s \approx V$$

where $(V_j; M_{j-1}, M_j)$ is an adjoint elementary cobordism with index $p_j|q_j$, such that

$$0|0 \leq p_1|q_1 \leq p_2|q_2 \leq \cdots \leq p_s|q_s \leq m+1|n+1$$

and $M_0 = N_0$, $M_k = N_1$. In (1), the symbol \approx denotes homeomorphism.

Proof. Let M be a quantum supermanifold of dimension $(m|n)$ and let us consider an embedding

$$\phi : \hat{S}^{p|q} \times \hat{D}^{m-p|n-q} \rightarrow M.$$

Let us consider the following $(m|n)$ -dimensional quantum supermanifold

$$M' \equiv \left(M \setminus \text{int}(\phi(\hat{S}^{p|q} \times \hat{D}^{m-p|n-q})) \right) \bigcup_{\phi(\hat{S}^{p|q} \times \hat{S}^{m-p-1|n-q-1})} \left(\hat{D}^{p+1|q+1} \times \hat{S}^{m-p-1|n-q-1} \right).$$

⁵Let us recall that V such that $\partial V = N_0 \sqcup N_1$, is called *quantum non-linear propagator* between the two Cauchy data N_0 and N_1 . This is the non-linear extension of the concepts of quantum propagator, usually used to quantize a classical field theory. For more details see Refs. [79, 81, 82].

We say that M' is obtained from M by a $(p|q)$ -surgery, i.e., cutting out $\text{int}(\hat{S}^{p|q} \times \hat{D}^{m-p|n-q})$ and gluing in $\hat{D}^{p+1|q+1} \times \hat{S}^{m-p-1|n-q-1}$. The process of surgery is related to cobordism and handle attaching ones. Let $(X, \partial X)$ be a $(m+1|n+1)$ -dimensional quantum supermanifold with boundary ∂X , and let

$$\phi : \hat{S}^{p|q} \times \hat{D}^{m-p|n-q} \rightarrow \partial X \equiv M$$

be an embedding. Set

$$X' \equiv X \bigcup_{\phi(\hat{S}^{p-1|q-1} \times \hat{D}^{m-p+1|n-q+1})} (\hat{D}^{p|q} \times \hat{D}^{m-p+1|n-q+1}).$$

We call X' obtained from X by *attaching a $(m+1|n+1)$ -dimensional quantum superhandle of index $p|q$* . One has

$$M' \equiv \partial X' = (\partial X \setminus \text{int}(\phi(\hat{S}^{p-1|q-1} \times \hat{D}^{m-p+1|n-q+1}))) \bigcup_{\phi(\hat{S}^{p-1|q-1} \times \hat{S}^{m-p|n-q})} (\hat{D}^{p|q} \times \hat{S}^{m-p|n-q}).$$

We say that M' is obtained from M by a quantum $(p-1|q-1)$ -surgery. The surgery allows us to obtain a cobordism

$$W \equiv (M \times \hat{D}^{1|1}) \bigcup_{\phi(\hat{S}^{p-1|q-1} \times \hat{D}^{m-p+1|n-q+1} \times (1,1))} (\hat{D}^{p|q} \times \hat{D}^{m-p+1|n-q+1})$$

with $\partial W = M \sqcup M'$.⁶

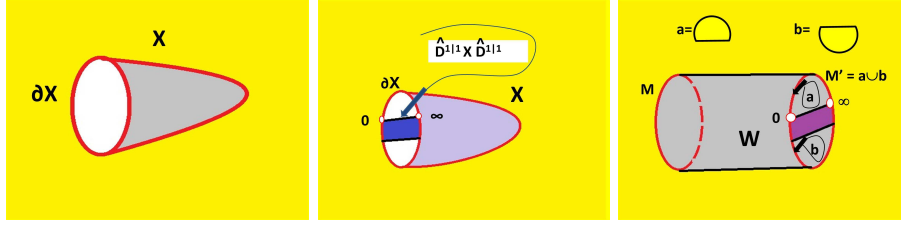


FIG. 1. Relation between attaching quantum superhandles and cobordisms on non-closed quantum supermanifold. The picture is made taking a $(2|2)$ -dimensional quantum supermanifold X with boundary ∂X . Then $X' \equiv X \bigcup_{\hat{S}^{0|0} \times \hat{D}^{1|1}} \hat{h}^{1|1}$ is obtained from X by attaching on the boundary $M \equiv \partial X$, a $(2|2)$ -dimensional quantum superhandle of index $1|1$, $\hat{h}^{1|1} = \hat{D}^{1|1} \times \hat{D}^{1|1}$. For the boundary $M' \equiv \partial X'$ one has $M' = (\partial X \setminus \text{int}(\hat{S}^{0|0} \times \hat{D}^{1|1})) \bigcup_{\hat{S}^{0|0} \times \hat{S}^{0|0}} \hat{D}^{1|1} \times \hat{S}^{0|0}$. This is a disjoint union of two of quantum supercircles. Then the $(2|2)$ -dimensional quantum supermanifold $W = (M \times \hat{D}^{1|1}) \bigcup_{\hat{S}^{0|0} \times \hat{D}^{1|1} \times \{\infty\}} \hat{D}^{1|1} \times \hat{D}^{1|1}$ is the bordism obtained by attaching to $M \times \hat{D}^{1|1}$ the $(2|2)$ -dimensional quantum superhandle of index $1|1$: $\hat{h}^{1|1} = \hat{D}^{1|1} \times \hat{D}^{1|1}$. In fact, one has $\partial W = M \sqcup M'$.

⁶In order to fix ideas we report in Fig. 2 some examples in dimension $(m|n) = (1|1), (2|2)$, showing the relation between attaching quantum superhandles and cobordism. In Fig. 1 is represented the case of a $(2|2)$ -dimensional quantum supermanifold X with boundary ∂X . Let us recall that $\hat{S}^{m|n} \equiv A^{m|n} \bigcup \{\infty\}$ is the $(m|n)$ -dimensional quantum supersphere and $\hat{D}^{m|n} \subset \hat{S}^{m|n}$, such that $\partial \hat{D}^{m|n} \cong \hat{S}^{m-1|n-1}$ is the quantum supermanifold called quantum $(m|n)$ -dimensional superdisk. One has $A^{0|0} = A^0 = \{0\}$ and $\partial \hat{D}^{1|1} \cong \hat{S}^{0|0} = \{0\} \bigcup \{\infty\}$. (For more information about see [79].)

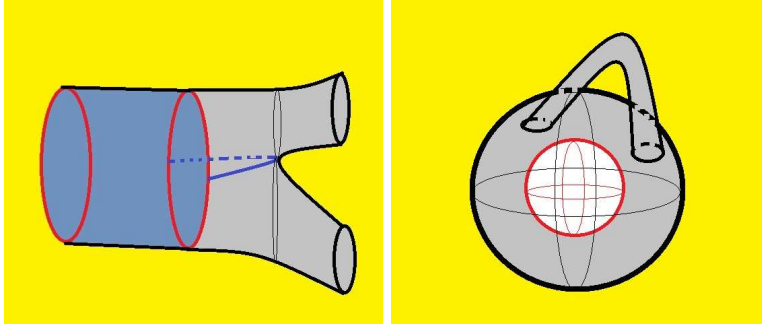


FIG. 2. Relation between attaching quantum superhandles and cobordisms. In the figure on the left, $\dim M = 1|1 = \dim M'$, $M = \hat{S}^{1|1}$ and $M' = M \cup_{\hat{S}^{0|0}=\{0,\infty\}} (\hat{D}^{1|1} \times \hat{S}^{0|0})$. The $(2|2)$ -dimensional quantum supermanifold blue colored is $M \times \hat{D}^{1|1}$. W is the $(2|2)$ -dimensional quantum supermanifold (blue-grey colored) bording M and M' , $\partial W = M \sqcup M'$. In the figure on the right, $\dim M = (2|2) = \dim M'$, $M = \hat{S}^{2|2}$ and $M' = M \cup_{\hat{S}^{0|0} \times \hat{S}^{1|1}} (\hat{D}^{1|1} \times \hat{S}^{1|1}) = M \cup (\hat{D}^{1|1} \times \hat{S}^{1|1})$. $W = (M \times \hat{D}^{1|1}) \cup_{\hat{S}^{0|0} \times \hat{D}^{2|2} \times \{\infty\}} \hat{D}^{1|1} \times \hat{D}^{2|2} = (M \times \hat{D}^{1|1}) \cup \hat{h}^{1|1}$ is the $(3|3)$ -dimensional quantum manifold, (grey colored), bording M and M' , $\partial W = M \sqcup M'$, where $\hat{h}^{1|1}$ is the $(3|3)$ -dimensional quantum superhandle of index $1|1$.

We shall use the following lemma.

Lemma 2.3. 1) *Every bordism $(W; M, M')$ in the category \mathfrak{Q}_S , with $\dim W = (m+1|n+1)$, $\dim M = \dim M' = m|n$, has a quantum superhandle decomposition as the union of a finite sequence*

$$(W; M, M') = (W_1; M_0, M_1) \bigcup (W_2; M_1, M_2) \bigcup \cdots (W_k; M_{k-1}, M_k)$$

of adjoint elementary conditions $(W_j; M_{j-1}, M_j)$ with index $p_j|q_j$, such that

$$0|0 \leq p_1|q_1 \leq p_2|q_2 \leq \cdots \leq p_s|q_s \leq m+1|n+1$$

and $M_0 = M$, $M_k = M'$.

2) *Closed $(m|n)$ -dimensional quantum super manifolds M , M' are cobordant iff M' can be obtained from M by a sequence of surgeries.*

3) *Every closed $(m|n)$ -dimensional quantum supermanifold can be obtained from \emptyset by attaching quantum superhandles.*





Proof. See Theorem 2.19 in [79].⁷ □

Let us remark that the proof of Lemma 2.3 is related to the quantum CW substitute structure of quantum nonlinear propagators, hence the relations considered in this lemma are to consider homeomorphisms. On the other hand the handle decomposition of the quantum nonlinear propagator V , is related also to the quantum cell

⁷Handle decomposition of bordism for manifolds has been introduced by Stephen Smale [96]. Lemma 2.3 generalizes to quantum supermanifolds an analogous result for commutative manifolds. However, the proof of Lemma 2.3 is not found on the Morse function, (see, e.g., [48]), but on the quantum CW-substitutes for quantum supermanifolds.

composition that always holds for a $(m+1|n+1)$ -dimensional quantum supermanifold, according to its quantum CW-structure. But the relation between a quantum superdisk $\hat{D}^{r|s}$ and a $(r|s)$ -dimensional quantum superhandle $\hat{h}^{p|q}$ of index $p|q$, is in general an homeomorphism $\hat{h}^{p|q} \approx \hat{D}^{p|q} \times \hat{D}^{r-p|s-q}$, that cannot be reduced to a diffeomorphism in the category \mathfrak{Q}_S .⁸ In Tab. 1 are reported some useful examples of quantum superhandle decompositions of quantum supermanifolds.

TAB. 1. Examples of quantum superhandle decompositions of quantum supermanifolds.

Name	Symbol	Handle-decomposition	Figures
Quantum $(m n)$ -supersphere	$\hat{S}^{m n}$	$\hat{h}^{0 0} \cup \hat{h}^{m n}$	
Quantum $(m+1 n+1)$ -supercobordism	$(\hat{D}^{m+1 n+1}; \emptyset, \hat{S}^{m n})$	$\hat{h}^{0 0}$	
Quantum $(2 2)$ -supertorus	$\hat{T}^{1 1} = \hat{S}^{1 1} \times \hat{S}^{1 1}$	$\hat{h}^{0 0} \cup \hat{h}^{1 1} \cup \hat{h}^{1 1} \cup \hat{h}^{2 2}$	
Quantum punctured Möbius band	$(\hat{M}_{ob} \setminus \hat{D}^{2 2}; \hat{S}^{1 1}, \hat{S}^{1 1})$	$\hat{S}^{1 1} \times \hat{D}^{1 1} \cup \hat{h}^{1 1}$	

Since V is a $(m+1|n+1)$ -dimensional bordism between $(m|n)$ -dimensional quantum supermanifolds N_0 and N_1 , we can determine its handle decomposition (1), that identifies intermediate final Cauchy quantum supermanifolds M_j that are obtained from the previous one M_{j-1} by means of an integral surgery. Let us emphasize that each M_j must necessarily be an integral quantum supermanifold having the same dimension of N_0 since it belongs to the same bordism class $[N_0]$. Furthermore, since each quantum supermanifold M_j is contained into V , it follows that M_j can be identified with a Cauchy $(m|n)$ -dimensional quantum supermanifold of \hat{E}_k , i.e. an admissible $(m|n)$ -dimensional quantum integral supermanifold of $\hat{E}_k \subset \hat{J}_{m+1|n+1}^k(W)$. Therefore M_j belongs to the same bordism class $[N_0] \in \Omega_{m|n}^{\hat{E}_k}$ of N_0 in \hat{E}_k . The situation is pictured in Fig. 3. \square

Definition 2.4 (Quantum exchangions and quantum virtual particles). *Let $\hat{E}_k \subset \hat{J}_{m+1|n+1}^k(W)$ be a quantum super PDE in the category \mathfrak{Q}_S and let $V \subset \hat{E}_k$ be a compact solution of a boundary value problem, i.e., $\partial V = N_0 \sqcup N_1$. We call quantum exchangions the quantum integral superhandle that are attached to obtain intermediate Cauchy manifolds $N_{j|j}$ defined in the proof of Theorem 2.2. We call quantum virtual particles.*

Remark 2.5. *Note the structural difference between quantum exchangions and virtual particles. In fact, the first are quantum $(m+1|n+1)$ -chains, hence having the same dimension of solutions of \hat{E}_k . Instead, the second ones are quantum*

⁸For example, let us consider the category of complex manifolds, say \mathfrak{C} . There the quantum algebra is the \mathbb{R} algebra of complex numbers \mathbb{C} , and the morphisms are holomorphic mappings, hence differentiable mappings, having \mathbb{C} -linear derivatives. Diffeomorphisms in \mathfrak{C} are usually called biholomorphic mappings. In fact, it is well known that do not exist biholomorphic mappings between an unit ball in \mathfrak{C} , (here identified with a *quantum m -disk*, \hat{D}^m), and the complex m -disk, D^m , (*generalized Poincaré's theorem* [40]).

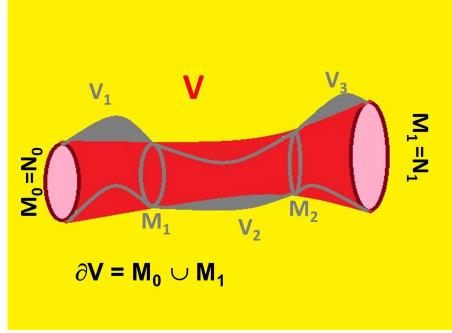


FIG. 3. Representation of quantum superhandle decomposition of quantum nonlinear propagator. The grey colored part denotes quantum superhandle decomposition of the bordism V (red colored). The grey bordisms identify the sections M_j on V , hence the adjoint elementary cobordisms V_j . For simplicity there is not represented any particular quantum superhandle, but only as particular bordisms are related to the quantum nonlinear propagator.

$(m|n)$ -chains, hence having the same dimension of the initial and final Cauchy data.⁹

Proposition 2.6. *A quantum exchangion does not change the integral bordism class of N_j with respect to N_0 one.*

Proof. In fact N_0 and M_j must necessarily belong to the same quantum integral bordism class in $\Omega_{m|n}^{\hat{E}_k}$ since $\partial V_1 = N_0 \sqcup M_2$ and $\partial V_2 = M_2 \sqcup M_3$, and so on. \square

Proposition 2.7. *Whether N_0 is the disjoint union of two (or more) components, e.g., $N_0 = a \sqcup b$, then a quantum exchangion can change the quantum numbers of a and b , even if the total quantum number of $a \sqcup b$ does not change.*

Proof. This follows directly from Proposition 2.6. \square

Definition 2.8 (Fundamental quantum particles). *For a quantum system, encoded by a quantum PDE $\hat{E}_k \subset \hat{J}_{m+1|n+1}^k(W)$, we define quantum $(m|n)$ -particles, admissible quantum integral $(m|n)$ -chains $N \subset \hat{E}_k \subset \hat{J}_{m+1|n+1}^k$.*

We call fundamental quantum $(m|n)$ -particles for the quantum PDE \hat{E}_k , ones that cannot be decomposed into other quantum particles.

Proposition 2.9. *The fundamental quantum $(m|n)$ -particles of quantum PDE $\hat{E}_k \subset \hat{J}_{m+1|n+1}^k(W)$, are identified with the bordism classes of the weak integral*

⁹It is useful to emphasize that quantum exchangions and quantum virtual particles, well interpret the meaning of "exchange particles" introduced in Feynman diagrams (1949) to represent particle interactions, like photons (electro-magnetic interactions), W and Z particles (weak interactions), gluons and gravitons (strong interactions). (See, e.g., Refs. [23, 38].) Furthermore, in the framework of Theorem 2.2 we understand also that *Regge trajectories* and *Regge resonances* (1959), introduced in the phenomenological theory of strong reactions by the pioneering works of T. Regge [92] (and principally developed also by R. Blanckebecker and M. L. Goldberger [7], G. F. Chew and S. C. Frautschi [14], V. N. Gribov [28] and G. Veneziano [100]), find their interpretation as quantum exchangions and quantum virtual particles.

bordism group $\Omega_{m|n,w}^{\hat{E}_k}$, (resp. singular integral bordism group $\Omega_{m|n,s}^{\hat{E}_k}$, resp. integral bordism group $\Omega_{m|n}^{\hat{E}_k}$). Then we will distinguish into weak-fundamental quantum $(m|n)$ -particles, singular-fundamental quantum $(m|n)$ -particles, resp. fundamental quantum $(m|n)$ -particles respectively, for the quantum PDE $\hat{E}_k \subset \hat{J}_{m+1|n+1}^k(W)$. Furthermore the exact commutative diagram (2) shows how such fundamental quantum particles are related.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & K_{m-1|n-1,w/(s,w)}^{\hat{E}_k} & \rightarrow & K_{m-1|n-1,w}^{\hat{E}_k} & \rightarrow & K_{m-1|n-1,s,w}^{\hat{E}_k} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (2) \quad 0 & \rightarrow & K_{m-1|n-1,s}^{\hat{E}_k} & \rightarrow & \Omega_{m-1|n-1}^{\hat{E}_k} & \rightarrow & \Omega_{m-1|n-1,s}^{\hat{E}_k} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \rightarrow & \Omega_{m-1|n-1,w}^{\hat{E}_k} & \rightarrow & \Omega_{m-1|n-1,w}^{\hat{E}_k} \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Proof. The proof follows directly from Definition 2.8, Theorem 2.2 and Proposition 3.3 in [74]. \square

Remark 2.10. Let us emphasize that the concept of fundamental quantum particles, is strictly related to the quantum (super) PDE \hat{E}_k , or in other words, to the quantum theory encoded by \hat{E}_k . Whether we aim to forget the framework defined by \hat{E}_k , and to discuss about fundamental quantum particles in general, we see that the unique fundamental one is \emptyset . In fact, according to Lemma 2.3(3), we can build any quantum particles just starting from \emptyset , and adding quantum handles or quantum-cells.¹⁰

Example 2.11 (Two-body high energy reactions). Let us consider the following typical two-body strong reactions: $\pi^- + p \rightarrow \pi^0 + n$, $\pi^+ + p \rightarrow \pi^+ + p$, $\pi^+ + p \rightarrow p + \pi^+$. The first reaction can be considered obtained with a quantum nonlinear propagator having an intermediate virtual neutral ρ meson, with charge exchange. In the second reaction the quantum nonlinear propagator has an intermediate virtual double electric charged particle and a neutral quantum pomeron p_{om} , quantum exchange $\hat{h}^{1|1}$, representing an elastic scattering. In the third reaction can be considered the quantum nonlinear propagator has an intermediate virtual neutron n , in backward

¹⁰From this geometric structural point of view, we clearly understand that also some particles, that are usually considered fundamental, cannot be considered so. For example electrons and quarks are not quantum fundamental particles whether considered as geometric objects. This agrees with experimental evidences that so-called "quasiparticles" with fractional charges, or with separated quantum numbers (e.g., decay $e^- \rightarrow \text{spinon} + \text{orbiton}$), were detected. (See, e.g. [24, 94].) Furthermore, it is important to distinguish the concept of quantum fundamental particle, from the one of quantum stable particle. Without any quantum theory, i.e., without any quantum (super) PDE, for the first the unique one is \emptyset , and for the second, it is a no sense. On the other hand, with respect to a quantum (super) PDE \hat{E}_k we can understand that the concept of stability is just related to the integral bordism groups of \hat{E}_k , hence from this point of view the concept of "fundamental particle" and "stable particle" become related each other one.

scattering. In other words, in all these reactions the final Cauchy data can be obtained from the initial one by means of two adjoint elementary bordisms.¹¹

Example 2.12 (Proton-proton chain reactions). *In the following we shall see that solutions, representing proton-proton chain reactions, typical in the Sun for the production of ${}^4_2\text{He}$, can be encoded with quantum nonlinear propagators, admitting handle decompositions. In general one can classify such reactions into five basic groups.*¹²

(I) Production of ${}^4_2\text{He}$ with intermediate particles $\{e^+, \nu_e, \gamma, {}^2_1\text{H} \equiv {}^2_1\text{D}, {}^3_2\text{He}\}$.¹³

$$(3) \quad \begin{cases} {}^1_1\text{H} + {}^1_1\text{H} \rightarrow e^+ + \nu_e + {}^2_1\text{H} + 0.42 \text{ MeV} \\ {}^2_1\text{D} + {}^1_1\text{H} \rightarrow {}^3_2\text{He} + \gamma + 5.49 \text{ MeV} \\ {}^3_2\text{H} + {}^3_2\text{He} \rightarrow 2 {}^1_1\text{H} + {}^4_2\text{He} + 12.86 \text{ MeV} \end{cases}$$

(II) Production of ${}^4_2\text{He}$ with intermediate particles $\{\nu_e, \gamma, {}^7_4\text{Be}, {}^7_3\text{Li}\}$.¹⁴

$$(4) \quad \begin{cases} {}^3_2\text{He} + {}^4_2\text{He} \rightarrow {}^7_4\text{Be} + \gamma \\ {}^7_4\text{Be} + e^- \rightarrow {}^7_3\text{Li} + \nu_e + 0.383 - 0.861 \text{ MeV} \\ {}^7_3\text{Li} + {}^1_1\text{H} \rightarrow 2 {}^4_2\text{He} \end{cases}$$

(III) Production of ${}^4_2\text{He}$ with intermediate particles $\{e^+, \nu_e, \gamma, {}^7_4\text{Be}, {}^8_4\text{Be}, {}^8_5\text{B}\}$.¹⁵

$$(5) \quad \begin{cases} {}^3_2\text{He} + {}^4_2\text{He} \rightarrow {}^7_4\text{Be} + \gamma \\ {}^7_4\text{Be} + {}^1_1\text{H} \rightarrow {}^8_5\text{B} + \gamma \\ {}^8_5\text{B} \rightarrow {}^8_4\text{Be} + e^+ + \nu_e \\ {}^8_4\text{Be} \rightarrow 2 {}^4_2\text{He} \end{cases}$$

(IV) Production of ${}^4_2\text{He}$ from the ${}^3_2\text{He}$ interaction with proton.

$$(6) \quad {}^3_2\text{He} + {}^1_1\text{H} \rightarrow {}^4_2\text{He} + e^+ + \nu_e + 18.8 \text{ MeV}.$$

(V) Production of deuterium from the electron capture by two protons.

$$(7) \quad {}^1_1\text{H} + e^- + {}^1_1\text{H} \rightarrow {}^2_1\text{D} + \nu_e.$$

In Fig. 4 we represent some solutions corresponding to reactions in (I). By using the definition introduced in [82], we can say that in both reactions there represented, the quantum nonlinear propagator V , is a quantum matter-solution. Furthermore, in the second reaction, the handle decomposition of V , identifies a Goldstone piece $V_\partial = V \cap (\widehat{\text{Goldstone}})$. This is the part arriving to γ .

¹¹The quantum virtual particles there involved can be considered generalizations in the geometric theory of quantum (super)PDEs, of objects called reggeons in analogous reactions considered in the framework of the phenomenological theory of strong reactions developed in the first two decades of the second half of the last century. (See, e.g., [22].) However, let us emphasize that *pomeron* is better represented as quantum exchangion than a quantum virtual particle. In fact, pomeron, as usually considered in the phenomenological theory of strong reactions, does not carry charge [22]. On the other hand in the elastic scattering $\pi^+ + p \rightarrow \pi^+ + p$ it is impossible that the quantum virtual particle should be neutral one.

¹²Let us recall the nucleus-notation ${}^A_Z\text{X}$, where X denotes the chemical symbol, $A = Z + N$ is the mass number, with Z the atomic number=number of protons and N the number of neutrons.

¹³In the Sun this chain reaction is dominant in the temperature range $10 - 14 \text{ MK}$. In presence of electrons, one has also the secondary reaction $e^+ + e^- \rightarrow 2\gamma + 1.02 \text{ MeV}$.

¹⁴In the Sun this chain reaction is dominant in the temperature range $14 - 23 \text{ MK}$.

¹⁵In the Sun this chain reaction is dominant in the temperatures that exceeds 23 MK .

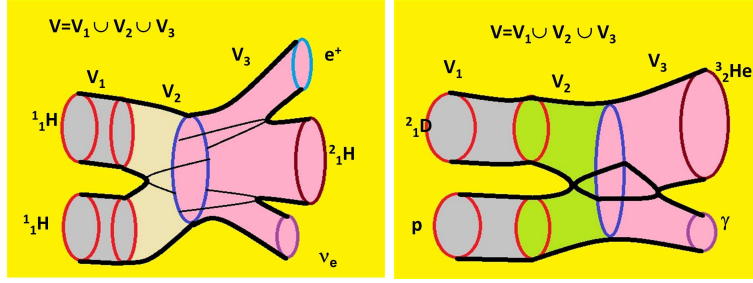


FIG. 4. Representation of some solutions of $p + p$ reaction chain. From left to right. ${}^1_1\text{H} + {}^1_1\text{H} \rightarrow e^+ + \nu_e + {}^2_1\text{H}$. Here the quantum nonlinear propagator V has a three step decomposition: $V_1 \cup V_2 \cup V_3$. Furthermore there is a virtual quantum particle M_3 with a double electric charge. One has $V_1 = M_0 \times \hat{D}^{1|1}$, $V_2 = \hat{h}^{1|1}$ and $V_3 = \hat{h}^{1|1} \cup \hat{h}^{1|1}$. The final Cauchy data is represented by three disjoint supermanifolds. In the reaction ${}^2_1\text{D} + {}^1_1\text{H} \rightarrow {}^3_2\text{He} + \gamma$, the quantum nonlinear propagator V has a three step decomposition too: $V_1 \cup V_2 \cup V_3$, but $V_1 = M_0 \times \hat{D}^{1|1}$, $V_2 = \hat{h}^{1|1}$ and $V_3 = \hat{h}^{1|1}$. There is a double electric charged virtual quantum particle M_3 , and the final Cauchy data is represented by two disjoint quantum supermanifolds.

3. Surgery in Quantum Super Yang-Mills PDEs

Let us emphasize, now, that in the reactions considered in the above section, we have not really specified the contribution of a specific quantum PDE, i.e., we have not considered some specific integral bordism group. On the other hand, without the introduction of these fundamental structures, what we can do is reproduce some phenomenological theory like dispersion relations and Regge models that cannot be but predictive dynamical theories. Therefore to dynamically encode proton-proton reactions we shall use the quantum super Yang-Mills PDE, and to solve suitable boundary value problems. We have formulated this general theory in some of our previous works [81, 82]. In the following we shall consider some particular applications.

Definition 3.1 (Quantum scattering processes in (\widehat{YM})). *We call quantum scattering process in (\widehat{YM}) any boundary value problem where are fixed two disjoint Cauchy data $N_0, N_1 \in (\widehat{YM})$. The first is called initial Cauchy data and the second the final Cauchy data. A solution of a quantum scattering process in (\widehat{YM}) is any quantum non linear propagator $V \subset (\widehat{YM})$, such that $\partial V = N_0 \sqcup N_1$, if $\partial N_0 = \partial N_1 = \emptyset$, otherwise $\partial V = N_0 \sqcup P \sqcup N_1$, with $P \subset (\widehat{YM})$, integral manifold such that $\partial P = \partial N_0 \sqcup \partial N_1$.*

Theorem 3.2 (Quantum scattering processes in (\widehat{YM}) on 4-dimensional quantum super Minkowskian manifold). *Let us assume that the base manifold M of (\widehat{YM}) is a $(4|4)$ -dimensional quantum super Minkowski manifold. Then (\widehat{YM}) is a quantum extended crystal PDE. Furthermore, if we consider admissible only integral boundary manifolds, with orientable classic limit, and with zero characteristic*

quantum supernumbers, (full admissibility hypothesis), one has: $\widehat{\Omega_{3|3}^{(YM)}} = 0$, and $\widehat{(YM)}$ becomes a quantum 0-crystal super PDE. Hence we get existence of global Q_w^∞ solutions for any boundary condition of class Q_w^∞ .

Elementary quantum nonlinear propagators V are $(4|4)$ -dimensional quantum supermanifolds with boundary $\partial V = N_0 \cup P \cup N_1$, such that V is homeomorphic to $\hat{D}^{4|4}$. One can classify V on the ground of its boundary ∂V . This can be diffeomorphic to a quantum exotic supersphere $\hat{\Sigma}^{3|3}$ or to a quantum supersphere $\hat{S}^{3|3}$.¹⁶

Proof. In $D = 4$, the usual N -supersymmetric extension \mathfrak{g} of the Poincaré algebra $\mathfrak{p} = \mathfrak{so}(1, 3) \oplus \mathfrak{t}$, is a \mathbb{Z}_2 -graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with a graded Lie bracket, such that $\mathfrak{g}_0 = \mathfrak{p} \oplus \mathfrak{b}$, where \mathfrak{b} is a reductive Lie algebra, such that its self-adjoint part is the tangent space to a real compact Lie group.¹⁷ Furthermore $\mathfrak{g}_1 = (\frac{1}{2}, 0) \otimes \mathfrak{s} \oplus (0, \frac{1}{2}) \otimes \mathfrak{s}^*$, where $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are specific representations of the Poincaré algebra. Both components are conjugate to each other under the $*$ conjugation. \mathfrak{s} is a N -dimensional complex representation of \mathfrak{b} and \mathfrak{s}^* its dual representation.¹⁸ Note also that the Lie bracket for the odd part is usually denoted by $\{, \}$ in theoretical physics. Then with such a notation one has

$$(8) \quad \{Q_\alpha^i, Q_\beta^j\} = \delta^{ij}(\gamma^\mu C)_{\alpha\beta} P_\mu + U^{ij}(C)_{\alpha\beta} + V^{ij}(C\gamma_5)_{\alpha\beta}$$

where $U^{ij} = -U^{ji}$, $V^{ij} = -V^{ji}$ are the $(N-1)N$ central charges, C is the (antisymmetric) charge conjugation matrix, $(Q_\alpha^i)_{i=1,\dots,N}$ are the N Majorana spinor supersymmetry charge generators. The dynamical components $\hat{\mu}^i$, $i = 1, \dots, N$, of the quantum fundamental field, corresponding to the generators Q^i , are called *quantum gravitinos*. So in a quantum N -SG-Yang-Mills PDE, one distinguishes N quantum gravitino types, (and $(N-1)N$ central charges).

Then a quantum superextension of \mathfrak{g} is $A \otimes_{\mathbb{R}} \mathfrak{g}$, where A is a quantum superalgebra. This can be taken $A \subseteq L(\mathcal{H})$, where \mathcal{H} is a super-Hilbert space. (See also Refs.[74].) In Tab. 2 are reported supersymmetric semi-simple tensor extensions of

TABLE 2. Supersymmetric semi-simple tensor extension Poincaré algebra in $D = 4$.

$[J_{\alpha\beta}, J_{\gamma\delta}] = \eta_{\beta\gamma} J_{\alpha\delta} + \eta_{\alpha\delta} J_{\beta\gamma} - \eta_{\alpha\gamma} J_{\beta\delta} - \eta_{\beta\delta} J_{\alpha\gamma}, \quad [P_\alpha, P_\beta] = c Z_{\alpha\beta}$
$[J_{\alpha\beta}, P_\gamma] = \eta_{\beta\gamma} P_\alpha - \eta_{\alpha\gamma} P_\beta, \quad [J_{\alpha\beta}, Z_{\gamma\delta}] = \eta_{\alpha\delta} Z_{\beta\gamma} + \eta_{\beta\gamma} Z_{\alpha\delta} - \eta_{\alpha\gamma} Z_{\beta\delta} - \eta_{\beta\delta} Z_{\alpha\gamma}$
$[Z_{\alpha\beta}, P_\gamma] = \frac{4a^2}{c}(\eta_{\beta\gamma} P_\alpha - \eta_{\alpha\gamma} P_\beta), \quad [Z_{\alpha\beta}, Z_{\gamma\delta}] = \frac{4a^2}{c}(\eta_{\alpha\delta} Z_{\beta\gamma} + \eta_{\beta\gamma} Z_{\alpha\delta} - \eta_{\alpha\gamma} Z_{\beta\delta} - \eta_{\beta\delta} Z_{\alpha\gamma})$
$[J_{\alpha\beta}, Q_\gamma] = -(\sigma_{\alpha\beta} Q)_\gamma, \quad [P_\alpha, Q_\gamma] = a(\gamma_\alpha Q)_\gamma, \quad [Z_{\alpha\beta}, Q_\gamma] = -\frac{4a^2}{c}(\sigma_{\alpha\beta} Q)_\gamma$
$[Q_\alpha, Q_\beta] = -b[\frac{2a}{c}(\gamma^\delta C)_{\alpha\beta} P_\delta + (\sigma^{\gamma\delta} C)_{\alpha\beta} Z_{\gamma\delta}]$

Poincaré algebra in $D = 4$ too. There a , b and c are constants. This algebra admits the following splitting: $\mathfrak{so}(3, 1) \oplus \mathfrak{osp}(1, 4)$, where $\mathfrak{so}(3, 1)$ is the 4-dimensional Lorentz algebra and $\mathfrak{osp}(1, 4)$ is the orthosymplectic algebra. Then, by considering

¹⁶In general quantum nonlinear propagators are not elementary ones, but can be decomposed in elementary ones.

¹⁷A *reductive* Lie algebra is the sum of a semisimple and an abelian Lie algebra. Since a *semisimple* Lie algebra is the direct sum of simple algebras, i.e., non-abelian Lie algebras, \mathfrak{l}_i , where the only ideals are $\{0\}$ and $\{\mathfrak{l}_i\}$, it follows that \mathfrak{b} can be represented in the form $\mathfrak{b} = \mathfrak{a} \oplus \sum_i \mathfrak{l}_i$.

¹⁸If $\rho : \mathfrak{g} \rightarrow L(V)$ is a representation of Lie algebra, its dual $\bar{\rho} : \mathfrak{g} \rightarrow L(\bar{V})$, working on the dual space \bar{V} , is defined by $\bar{\rho}(u) = -\rho(u)^*$, $\forall u \in \mathfrak{g}$.

the quantum superextension $A \otimes_{\mathbb{R}} [\mathfrak{so}(3, 1) \oplus \mathfrak{osp}(1, 4)]$, where A is a quantum superalgebra, and with respect to the splitting $\hat{\mu} = {}_{\otimes}\hat{\mu} + {}_{\oplus}\hat{\mu} + {}_{\boxtimes}\hat{\mu}$ of the fundamental field $\hat{\mu}$, we get:

$$(9) \quad \begin{cases} {}_{\otimes}\hat{\mu} = P_{\alpha} \hat{\theta}_{\gamma}^{\alpha} dx^{\gamma} \\ {}_{\oplus}\hat{\mu} = J_{\alpha\beta} \hat{\omega}_{\gamma}^{\alpha\beta} dx^{\gamma} \\ {}_{\boxtimes}\hat{\mu} = [\bar{Z}_{\alpha\beta} \hat{A}_{\gamma}^{\alpha\beta} + Q_{\alpha i} \phi_{\gamma}^{\alpha i}] dx^{\gamma}. \end{cases}$$

The dynamic equation are resumed in Tab. 3.

TABLE 3. Local expression of $\widehat{(YM)} \subset J\hat{D}^2(W)$ and Bianchi identity $(B) \subset J\hat{D}^2(W)$.

(Field equations)	$E_K^A \equiv -(\partial_B \cdot \hat{R}_K^{BA}) + [\hat{C}_{KR}^H \hat{\mu}_C^R, \hat{R}_H^{[AC]}]_+ = 0$	$\widehat{(YM)}$
$\hat{R}_{A_1 A_2}^K =$	$(\partial X_{A_1} \cdot \hat{\mu}_{A_2}^K) + \frac{1}{2} \hat{C}_{IJ}^K [\hat{\mu}_{A_1}^I, \hat{\mu}_{A_2}^J]_+$	(Fields)
(Bianchi identities)	$B_{HA_1 A_2}^K \equiv (\partial X_H \cdot \hat{R}_{A_1 A_2}^K) + \frac{1}{2} \hat{C}_{IJ}^K [\hat{\mu}_H^I, \hat{R}_{A_1 A_2}^J]_+ = 0$	(B)

$\hat{R}_{A_1 A_2}^K : \Omega_1 \subset J\hat{D}(W) \rightarrow \hat{A}; \quad B_{HA_1 A_2}^K : \Omega_2 \subset J\hat{D}^2(W) \rightarrow \hat{A}; \quad E_K^A : \Omega_2 \subset J\hat{D}^2(W) \rightarrow \hat{A}.$

We call *quantum graviton* a quantum metric \hat{g} obtained by a solution $\hat{\mu}$ of $\widehat{(YM)}$, via the corresponding quantum vierbein.

Since $H_3(M; \mathbb{K}) = 0$, we get

$$(10) \quad \begin{cases} \Omega_{3|3,w}^{(\widehat{YM})} & \cong \Omega_{3|3,s}^{(\widehat{YM})} \\ & \cong A_0 \otimes_{\mathbb{K}} H_3(W; \mathbb{K}) \oplus A_1 \otimes_{\mathbb{K}} H_3(W; \mathbb{K}) \\ & \cong A_0 \otimes_{\mathbb{K}} H_3(M; \mathbb{K}) \oplus A_1 \otimes_{\mathbb{K}} H_3(M; \mathbb{K}) = 0 \end{cases}$$

So $\widehat{(YM)}$ is a quantum extended crystal super PDE. However, in general, $\widehat{(YM)}$ is not a quantum 0-crystal super PDE. In fact one has the following short exact sequence

$$0 \longrightarrow \ker(j) \longrightarrow \Omega_{3|3}^{(\widehat{YM})} \xrightarrow{j} \Omega_{3|3,s}^{(\widehat{YM})} \longrightarrow 0$$

hence $\Omega_{3|3}^{(\widehat{YM})} \cong \ker(j) \neq 0$. Note that $\ker(j)$ is made by $[N] \in \Omega_{3|3}^{(\widehat{YM})}$ such that $N = \partial V$, where V is some $(4|4)$ -dimensional quantum supermanifold identified with a submanifold of $\hat{J}_{4|4}^2(W)$. However, if we consider admissible only integral boundary manifolds, with orientable classic limit, and with zero characteristic quantum supernumbers, (*full admissibility hypothesis*), one has: $\Omega_{3|3}^{(\widehat{YM})} = 0$, and $\widehat{(YM)}$ becomes a quantum 0-crystal super PDE. Hence we get existence of global Q_w^{∞} solutions for any boundary condition of class Q_w^{∞} .

Then we get the exact commutative diagram (11). (For notation see [81, 82].)

$$(11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & K_{3|3;2}^{(\widehat{YM})} & \longrightarrow & \Omega_{3|3}^{(\widehat{YM})} & \longrightarrow & \Omega_6^{(\widehat{YM})} \longrightarrow 0 \\ & & & & & \searrow & \downarrow \\ & & & & & & \Omega_6^{\uparrow} \\ 0 & \longrightarrow & K_6^{\uparrow} & \longrightarrow & \Omega_6^{\uparrow} & \longrightarrow & \Omega_6 \longrightarrow 0 \end{array}$$

Taking into account the result by Thom on the unoriented cobordism groups [99], we can calculate $\Omega_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then, we can represent Ω_6 as a subgroup of a 3-dimensional crystallographic group type $[G(3)]$. In fact, we can consider the amalgamated subgroup $D_2 \times \mathbb{Z}_2 \star_{D_2} D_4$, and monomorphism $\Omega_6 \rightarrow D_2 \times \mathbb{Z}_2 \star_{D_2} D_4$, given by $(a, b, c) \mapsto (a, b, b, c)$. Alternatively we can consider also $\Omega_6 \rightarrow D_4 \star_{D_2} D_4$. (See Appendix C in [80] for amalgamated subgroups of $[G(3)]$.) In any case the crystallographic dimension of (YM) is 3 and the crystallographic space group type are D_{2d} or D_{4h} belonging to the tetragonal syngony.

• If the initial and final Cauchy data are non-closed compact bounded 3|3-dimensional quantum supermanifolds, we can consider the exact commutative diagram reported in (12). There $\bar{B}_{\bullet|\bullet}(\hat{E}_k; A) = \ker(\partial|_{\bar{C}_{\bullet|\bullet}(\hat{E}_k; A)})$, $\bar{Z}_{\bullet|\bullet}(\hat{E}_k; A) = \text{im}(\partial|_{\bar{C}_{\bullet|\bullet}(\hat{E}_k; A)})$, $\bar{H}_{\bullet|\bullet}(\hat{E}_k; A) = \bar{Z}_{\bullet|\bullet}(\hat{E}_k; A)/\bar{B}_{\bullet|\bullet}(\hat{E}_k; A)$. Furthermore,

$$\left\{ \begin{array}{l} b \in [a] \in \bar{B}or_{\bullet|\bullet}(\hat{E}_k; A) \Rightarrow a - b = \partial c, \quad c \in \bar{C}_{\bullet|\bullet}(\hat{E}_k; A), \\ b \in [a] \in \bar{C}yc_{\bullet|\bullet}(\hat{E}_k; A) \Rightarrow \partial(a - b) = 0, \\ b \in [a] \in {}^A\bar{\Omega}_{\bullet|\bullet, s}(\hat{E}_k) \Rightarrow \left\{ \begin{array}{l} \partial a = \partial b = 0 \\ a - b = \partial c, \quad c \in \bar{C}_{\bullet|\bullet}(\hat{E}_k; A) \end{array} \right\} \end{array} \right\}.$$

It follows the canonical isomorphism: ${}^A\bar{\Omega}_{\bullet|\bullet, s}(\hat{E}_k) \cong \bar{H}_{\bullet|\bullet}(\hat{E}_k; A)$. As $\bar{C}_{\bullet|\bullet}(\hat{E}_k; A)$ is a free two-sided projective A -module, one has the unnatural isomorphism:

$$\bar{B}or_{\bullet|\bullet}(\hat{E}_k; A) \cong {}^A\bar{\Omega}_{\bullet|\bullet, s}(\hat{E}_k) \bigoplus \bar{C}yc_{\bullet|\bullet}(\hat{E}_k; A).$$

Then $a \equiv N_0 \cup P$ and $b \equiv N_1$ belong to the same bordism group. More precisely one has

$$b \in [a] \in \bar{B}or_{3|3}^{\hat{E}_k}, \Leftrightarrow a - b = \partial c, \quad c \equiv V,$$

such that, $\partial V = N_0 \cup P \cup N_1$. In general one has the following relation with the (closed) bordism group $\Omega_{3|3}^{\hat{E}_k}$:

$$\bar{B}or_{3|3}^{\hat{E}_k} \cong \Omega_{3|3}^{\hat{E}_k} \bigoplus \bar{C}yc_{3|3}^{\hat{E}_k}$$

where the cyclism group $\bar{C}yc_{3|3}^{\hat{E}_k}$, is defined by the condition $b \in [a] \in \bar{C}yc_{3|3}^{\hat{E}_k}$ iff $\partial(a - b) = 0$. Since, under the full admissibility hypothesis, one has $\Omega_{3|3}^{\hat{E}_k} = 0$, we get $\bar{B}or_{3|3}^{\hat{E}_k} \cong \bar{C}yc_{3|3}^{\hat{E}_k}$. Therefore, we can say that inequivalent quantum particles in \hat{E}_k , are identified with (3|3)-dimensional bounded compact quantum supermanifolds, Cauchy data in \hat{E}_k , that are in correspondence one-to-one with bordism classes in $\bar{B}or_{3|3}^{\hat{E}_k} \cong \bar{C}yc_{3|3}^{\hat{E}_k}$. Therefore, we get that the group of such fundamental particles are identified with

$$\bar{C}_{3|3}(\hat{E}_k; A)/\bar{B}_{3|3}(\hat{E}_k; A) \cong \bar{C}_{3|3}(\hat{E}_k; A)/\bar{Z}_{3|3}(\hat{E}_k; A).$$

(12)

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{B}_{\bullet|\bullet}(\hat{E}_k; A) & \longrightarrow & \bar{Z}_{\bullet|\bullet}(\hat{E}_k; A) & \longrightarrow & \bar{H}_{\bullet|\bullet}(\hat{E}_k; A) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \bar{C}_{\bullet|\bullet}(\hat{E}_k; A) & \xlongequal{\quad} & \bar{C}_{\bullet|\bullet}(\hat{E}_k; A) & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}^A\bar{\Omega}_{\bullet|\bullet,s}(\hat{E}_k) & \longrightarrow & \bar{B}or_{\bullet|\bullet}(\hat{E}_k; A) & \longrightarrow & \bar{C}yc_{\bullet|\bullet}(\hat{E}_k; A) \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

• Above results mean that all $(3|3)$ -dimensional closed integral quantum supermanifolds $X \subset \hat{E}_k$ are boundary of some $4|4$ -dimensional integral quantum supermanifold $V \subset \hat{E}_k$, i.e., $X = \partial V$. In other words, fixing two Cauchy data N_0 and N_1 in \hat{E}_k , we can consider a quantum non-linear propagator V between them, identified by means of a $(3|3)$ -dimensional closed compact integral quantum supermanifold $X \subset \hat{E}_k$, such that $X = N_0 \cup P \cup N_1$. Therefore a quantum non-linear propagator between N_0 and N_1 can be identified with a connected $(3|3)$ -dimensional closed compact integral quantum supermanifold $X \subset \hat{E}_k$ homeomorphic to $N_0 \cup P \cup N_1$. Since $\Omega_{3|3} = 0$, it follows that we can represent such quantum non-linear propagators, with a $(3|3)$ -dimensional quantum supersphere $\hat{S}^{3|3}$. Taking into account results contained in [87], we can also distinguish between them $(3|3)$ -dimensional quantum exotic superspheres $\hat{\Sigma}^{3|3}$. These last are classified with respect to the equivalence relation induced by quantum diffeomorphisms.

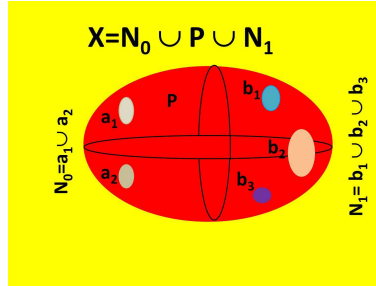


FIG. 5. Representation of an elementary quantum non-linear propagator V , identified by means of a $(3|3)$ -dimensional quantum supersphere $X = N_0 \cup P \cup N_1$, such that $\partial V = X$.

In Fig. 5 is represented the boundary $X = \partial V$, of an elementary quantum non-linear propagator V , represented as a $(3|3)$ -dimensional quantum supersphere $\hat{S}^{3|3}$. Therefore, such quantum non-linear propagators are homeomorphic to a $(4|4)$ -dimensional quantum superdisk $\hat{D}^{4|4}$. Fixing a $(3|3)$ -dimensional quantum (exotic) supersphere $X \approx \hat{\Sigma}^{3|3} \subset \hat{E}_k$, we can embed there $(3|3)$ -dimensional quantum superdisks a_i , $i = 1, \dots, p$ and b_j , $j = 1, \dots, q$, and consider the $(4|4)$ -dimensional quantum integral submanifold $V \subset \hat{E}_k$, identified by a fixed boundary X . Then we

can consider V the quantum non-linear propagator between $N_0 \equiv \bigcup_{1 \leq i \leq p} a_i$ and $N_1 \equiv \bigcup_{1 \leq j \leq q} b_j$, such that $\partial V = N_0 \cup P \cup N_1$, with $P \equiv \overline{X \setminus (N_0 \cup N_1)}$.

• It is important to remark that above quantum non-linear propagators, V , cannot be considered regular solutions in the equation $\widehat{(YM)} \subset J\hat{D}^2(W)$, but are necessarily singular solutions there. However, by considering the natural embedding $J\hat{D}^2(W) \hookrightarrow \hat{J}_{4|4}^2(W)$ we can consider V as regular solution of $\widehat{(YM)} \subset \hat{J}_{4|4}^2(W)$. But, whether N_0 is not diffeomorphic to N_1 , we cannot say that V is diffeomorphic to $N_0 \times \hat{D}^{1|1}$. Hence, the integrable full-quantum vector field $\zeta : V \rightarrow \hat{TV}$, relating N_0 to N_1 , must necessarily be singular one.

• For any quantum conservation law α of $\widehat{(YM)}$ we have $\langle \alpha, X \rangle = 0$, if $X = \partial V$, where V is a quantum nonlinear propagator between N_0 and N_1 . Therefore we get the relation (13) between quantum numbers induced by the quantum conservation laws:¹⁹

$$(13) \quad \langle \alpha, N_0 \rangle - \langle \alpha, N_1 \rangle = - \langle \alpha, P \rangle \in B.$$

Therefore, identifying quantum nonlinear propagators, for N_0 and N_1 , with $(4|4)$ -dimensional quantum integral superdisks $\hat{D}^{4|4} \subset \hat{E}_k$, we get that the relation (13) between the corresponding quantum conservation supernumbers, gives

$$P \approx \overline{\hat{S}^{4|4} \setminus (N_0 \cup N_1)}.$$

□

Similar considerations hold for the observed quantum non-linear propagators. (See also [81, 82].)

Theorem 3.3. *The observed dynamic equation $\widehat{(YM)}[i]$, by means of a quantum relativistic frame, is a quantum extended crystal super PDE. Moreover, under the full admissibility hypothesis, it becomes a quantum 0-crystal super PDE.*

Proof. The evaluation of $\widehat{(YM)}$ on a macroscopic shell $i(M_C) \subset M$ is given by the equations reported in Tab. 4.

TABLE 4. Local expression of $\widehat{(YM)}[i] \subset J\hat{D}^2(i^*W)$ and Bianchi identity $(B)[i] \subset J\hat{D}^2(i^*W)$.

(Observed Field Equations)	$(\partial_\alpha \cdot \tilde{R}^{K\alpha\beta}) + [\hat{C}_{IJ}^K \tilde{\mu}_\alpha^I, \tilde{R}^{J\alpha\beta}]_+ = 0$	$\widehat{(YM)}[i]$
	$\tilde{R}_{\alpha_1\alpha_2}^K = (\partial_{\xi_{[\alpha_1}} \cdot \tilde{\mu}_{\alpha_2]}^K) + \frac{1}{2} \hat{C}_{IJ}^K \tilde{\mu}_{[\alpha_2}^I \tilde{\mu}_{\alpha_1]}^J$	(Observed Fields)
(Observed Bianchi Identities)	$(\partial_{\xi_{[\gamma}} \cdot \tilde{R}_{\alpha_1\alpha_2]}^K) + \frac{1}{2} \hat{C}_{IJ}^K \tilde{\mu}_{[\gamma}^I \tilde{R}_{\alpha_1\alpha_2]}^J = 0$	$(B)[i]$

This equation is also formally integrable and completely integrable. Furthermore, the 3-dimensional integral bordism group of $\widehat{(YM)}[i]$ and its infinity prolongation $\widehat{(YM)}[i]_{+\infty}$ are trivial, under the full admissibility hypothesis:

$$\Omega_3^{\widehat{(YM)}[i]} \cong \Omega_3^{\widehat{(YM)}[i]_{+\infty}} \cong 0.$$

¹⁹ In the particular case that $P = \emptyset$, i.e., $\partial N_0 = \partial N_1 = \emptyset$, then $\langle \alpha, N_0 \rangle = \langle \alpha, N_1 \rangle$.

So equation $(\widehat{YM})[i] \subset J\hat{D}^2(i^*W)$ becomes a quantum 0-crystal super PDE and it admits global (smooth) solutions for any fixed time-like 3-dimensional (smooth) boundary conditions. Whether $N_0 \not\cong N_1$, has as a consequence that, considering the analog boundary value problem in the observed PDE $(\widehat{YM})[i] \subset \hat{J}_3^2 W$, cannot exist an observed smooth solution V , representing an observed quantum non-linear propagator between N_0 and N_1 . The corresponding observed quantum nonlinear propagator must necessarily be singular one. \square

4. Some open problems in high energy physics solved

In this section we will use the geometric mathematical architecture of the algebraic topology of quantum super PDEs previously developed to answer to some important open problems in High Energy Physics.

• Let us, first, consider the following question: "*Does exist a linear dependence between square mass and spin in a solution of the quantum PDE (\widehat{YM}) ?*".²⁰ We shall see that in general such simple dependence for quantum solutions is not assured, but there is a very more complex relation between quantum mass and quantum spin.

Let us recall our some previous results about quantum torsion.

The quantum vierbein curvature ${}_{\otimes}\hat{R}$ identifies, by means of the quantum vierbein $\hat{\theta}$ a quantum field $\hat{S} : M \rightarrow Hom_Z(\dot{\Lambda}_0^2 M; TM)$, that we call *quantum torsion*, associated to $\hat{\mu}$. In quantum coordinates one can write

$$(14) \quad \hat{S} = \partial x_C \otimes \hat{S}_{AB}^C dx^A \wedge dx^B, \quad \hat{S}_{AB}^C = \hat{\theta}_{K\otimes}^C \hat{R}_{AB}^K.$$

Furthermore, with respect to a quantum relativistic frame $i : N \rightarrow M$, the quantum torsion \hat{S} identifies a A -valued $(1, 2)$ -tensor field on N , $\tilde{S} \equiv i^* \hat{S} : N \rightarrow A \otimes_{\mathbb{R}} \Lambda_2^0 N \otimes_{\mathbb{R}} TN$, that we call *quantum torsion of the observed solution*.

We say that an observed solution has a *quantum spin*, if the observed solution has an observed torsion

$$(15) \quad \tilde{S} \equiv i^* \hat{S} = \partial x_\gamma \otimes \sum_{0 \leq \alpha < \beta \leq 3} \tilde{S}_{\alpha\beta}^\gamma dx^\alpha \wedge dx^\beta : N \rightarrow A \otimes_{\mathbb{R}} N \otimes_{\mathbb{R}} \Lambda_2^0(N) \cong A \otimes_{\mathbb{R}} \Lambda_2^0(N) \otimes_{\mathbb{R}} TN$$

with $\tilde{S}_{\alpha\beta}^\gamma(p) = -\tilde{S}_{\beta\alpha}^\gamma(p) \in A$, $p \in N$, that satisfies the following conditions, (*quantum-spin-conditions*):

$$(16) \quad \left\{ \begin{array}{l} \tilde{S} = \tilde{s} \otimes \dot{\psi} \\ \tilde{s} = \sum_{0 \leq \alpha < \beta \leq 3} \tilde{s}_{\alpha\beta} dx^\alpha \wedge dx^\beta : N \rightarrow A \otimes_{\mathbb{R}} \Lambda_2^0 N, \\ \tilde{s}_{\alpha\beta}(p) = -\tilde{s}_{\beta\alpha}(p) \in A, p \in N, \\ \dot{\psi} \lrcorner \tilde{S} = 0 \end{array} \right\} \\ \Downarrow \\ \left\{ \begin{array}{l} \tilde{S}_{\alpha\beta}^\lambda = \tilde{s}_{\alpha\beta} \dot{\psi}^\lambda \\ \tilde{S}_{\alpha\beta}^\lambda \dot{\psi}^\alpha = 0 \end{array} \right\}.$$

²⁰This question arises from some semiclassical approaches that suggest to consider the following relation $J \simeq \alpha' M^2$, between angular momentum J and mass M , for a spherical rotating object. This is just the philosophy of so-called *Regge trajectories*. Such linear trajectories are used to interpret scattering amplitudes strong reactions and are usually seen as gluonic strings attached to quarks at the end points. However there are also more exotic points of view where instead are considered pion excitations of light hadrons. (See, e.g., [16].)

where $\dot{\psi}$ is the velocity field on N of the time-like foliation representing the quantum relativistic frame on N . When conditions (16) are satisfied, we say that the solution considered admits a *quantum spin-structure*, with respect to the quantum relativistic frame. We call \tilde{s} the *quantum 2-form spin* of the observed solution. Let $\{\xi^\alpha\}_{0 \leq \alpha \leq 3}$ be coordinates on N , adapted to the quantum relativistic frame. Then one has the following local representations:

$$(17) \quad \left\{ \tilde{s} = \tilde{s}_{ij} d\xi^i \wedge d\xi^j \right\}.$$

We define *quantum spin-vector-field* of the observed solution

$$(18) \quad \tilde{\underline{s}} = \langle \epsilon, \tilde{S} \rangle = [\epsilon_{\mu\nu\lambda\rho} \dot{\psi}^\mu \tilde{s}^{\nu\lambda}] d\xi^\rho \equiv \tilde{s}_\rho d\xi^\rho = \tilde{s}_k dx \xi^k \Rightarrow \tilde{s} = \partial \xi_i \tilde{s}_k g^{ki} = \partial \xi_i \tilde{s}^i$$

where $\epsilon_{\mu\nu\lambda\rho} = \sqrt{|g|} \delta_{\mu\nu\lambda\rho}^{0123}$ is the completely antisymmetric tensor density on N . One has $\tilde{s}^\rho(p) \in A$, $p \in N$. The classification of the observed solution on the ground of the spectrum of $|\tilde{s}|^2 \equiv \tilde{s}^\rho \tilde{s}_\rho$, and its (*quantum helicity*), i.e., component \tilde{s}_z , is reported in Tab. 5.

TABLE 5. Local quantum spectral-spin-classification of $(\widehat{YM})[i]$ solutions.

Definition	Name
$Sp(\tilde{s}(p) ^2) \subset \mathfrak{b} \equiv \{h^2 s(s+1) s \in \mathbb{N} \equiv \{0, 1, 2, \dots\}\}, \quad Sp(\tilde{s}_z(p)) \subset \mathfrak{c}$	bosonic-polarized
$Sp(\tilde{s}(p) ^2) \subset \mathfrak{f} \equiv \{h^2 s(s+1) s = \frac{2n+1}{2}, n \in \mathbb{N} \equiv \{0, 1, 2, \dots\}\}, \quad Sp(\tilde{s}_z(p)) \subset \mathfrak{c}$	fermionic-polarized
$Sp(\tilde{s}(p) ^2) \cap \mathfrak{b} = Sp(\tilde{s}(p) ^2) \cap \mathfrak{f} = \emptyset$	unpolarized
$Sp(\tilde{s}(p) ^2) \cap \mathfrak{b} \neq \emptyset, \text{ and/or } Sp(\tilde{s}(p) ^2) \cap \mathfrak{f} \neq \emptyset$	mixt-polarized

$|\tilde{s}(p)|^2 \equiv \tilde{s}^\rho(p) \tilde{s}_\rho(p) \in \hat{A}$, $p \in N$. $\mathfrak{c} \equiv \{hm_s | m_s = -s, -s+1, \dots, s-1, s\}$. \tilde{s}_z quantum helicity.
 s = spin quantum number; m_s = spin orientation quantum number.

(\widehat{YM}) is a functional stable quantum super PDE since it is completely integrable and formally integrable. (See Theorem 2.34 in [82]). For the same reason it admits $(\widehat{YM})_{+\infty} \subset J\hat{D}^\infty(W)$ like stable quantum extended crystal PDE. Furthermore, since its symbol \hat{g}_2 is not trivial, any global solution $V \subset (\widehat{YM})$ can be unstable, and the corresponding observed solution, can appear unstable in finite times. However, global smooth solution, result stable in finite times in $(\widehat{YM})_{+\infty}$.

Finally the asymptotic stability study of global solutions of (\widehat{YM}) , with respect to a quantum relativistic frame, can be performed by means of Theorem 2.46 in [82], since, for any section $s : M \rightarrow W$, on the fibers of $\hat{E}[s] \rightarrow M$ there exists a non-degenerate scalar product. In fact, $\hat{E}[s] \cong W$, as W is a vector bundle over M . Furthermore, for any section s , we can identify on M a non degenerate metric \hat{g} ,²¹ that beside the rigid metric \underline{g} on \mathfrak{g} , identifies a non-degenerate metric on each fiber $\hat{E}[s]_p \cong W_p = Hom_Z(T_p M; \mathfrak{g})$, $\forall p \in M$. In fact we get $\hat{\xi}(p) \cdot \hat{\xi}(p)' = \underline{g}_{KH} \hat{g}^{AB}(p) \xi_A^K(p) \otimes \xi_B^H(p) \in A$.

Solutions of (YM) that encode nuclear-charged plasmas, or nuclides, dynamics. These are described by solutions that, when observed by means of a quantum relativistic frame have at any $t \in T$, i.e., frame-proper time, compact sectional

²¹It should be more precise to denote \hat{g} with the symbol $\hat{g}[s]$, since it is identified by means of the section s .

support $B_t \subset N$. The *global mass* at the time t , i.e. the evaluation

$$m_t = \int_{B_t} m(t, \xi^k) \sqrt{\det(g_{ij})} d\xi^1 \wedge d\xi^2 \wedge d\xi^3$$

of such mass on the space-like section B_t , gives the global mass-contents of the nuclear-plasmas or nuclides, in their ground-eigen-states, at the proper time t . Whether such solutions are asymptotically stable, then they interpret the meaning of stable nuclear-plasmas or nuclides.

Lemma 4.1. *If a solution admits spin structure, then we get $|\tilde{S}|^2 = |\tilde{s}|^2$. Furthermore in a frame-adapted coordinates we can write*

$$(19) \quad \begin{cases} |\tilde{s}|^2 &= \tilde{s}_{\alpha\beta} \tilde{s}^{\alpha\beta} \dot{\psi}^\gamma \dot{\psi}_\gamma \\ &= \tilde{s}_{\alpha\beta} \tilde{s}^{\alpha\beta} \\ &= \tilde{\mu}_K^\gamma \otimes \tilde{R}_{\alpha\beta}^K \otimes \tilde{R}^{\alpha\beta}_{\bar{K}} \tilde{\mu}_{\bar{\gamma}}^{\bar{K}} = |\tilde{S}|^2. \end{cases}$$

Proof. Let us first recall that the quantum torsion $\hat{S} : M \rightarrow \text{Hom}_Z(\dot{\Lambda}_0^2 M; TM)$ and its dual $\hat{\underline{S}} : M \rightarrow \text{Hom}_Z(TM; \dot{\Lambda}_0^2 M)$ are defined, for any $p \in M$, by the compositions reported in (20).

$$(20) \quad \begin{array}{ccccc} \dot{\Lambda}_0^2(T_p M) & \xrightarrow{\otimes \hat{R}(p)} & \hat{\mathfrak{g}} & \xrightarrow{\hat{\theta}^{-1}} & T_p M \\ & \searrow \hat{S}(p) & & & \nearrow \hat{\underline{S}}(p) \\ T_p M & \xrightarrow{\hat{\theta}} & \hat{\mathfrak{g}} & \xrightarrow{\otimes \hat{R}(p)} & \dot{\Lambda}_0^2(T_p M) \end{array}$$

Then, by taking the pull-back with respect to the embedding $i : N \rightarrow M$, identified by the quantum relativistic frame, we get

$$|\tilde{S}|^2 = i^*(\hat{S} \hat{\underline{S}}) = \tilde{S}_{\alpha\beta}^\gamma \tilde{S}_\gamma^{\alpha\beta} = \tilde{\mu}_K^\gamma \otimes \tilde{R}_{\alpha\beta}^K \otimes \tilde{R}^{\alpha\beta}_{\bar{K}} \tilde{\mu}_{\bar{\gamma}}^{\bar{K}}.$$

Therefore, taking into account that in the case of solution with spin structure, one has $\tilde{S}_{\alpha\beta}^\gamma = \dot{\psi}_\gamma \tilde{s}^{\alpha\beta}$ and $\tilde{S}_\gamma^{\alpha\beta} = \dot{\psi}^\gamma \tilde{s}_{\alpha\beta}$, we get (19). \square

• The following shows how thermodynamic functions can be associated to solutions of (YM) .

Let \tilde{H} be the Hamiltonian corresponding to an observed solution of (YM) . Let us recall that \tilde{H} is a \hat{A} -valued function on the 4-dimensional space-time N , considered in the quantum relativistic frame. Let us denote by $E \in Sp(\tilde{H})$. If $N(E) = \text{tr } \delta(E - \tilde{H})$ denotes the degeneracy of E , let us define *local partition function* of the observed solution the Laplace transform of the degeneracy $N(E)$, with respect the spectrum $Sp(\tilde{H})$ of \tilde{H} . We get

$$(21) \quad \begin{cases} Z(\beta) &= \int_{Sp(\tilde{H})} e^{-\beta E} N(E) dE \\ &= \int_{Sp(\tilde{H})} e^{-\beta E} \text{tr } \delta(E - \tilde{H}) dE \\ &= \text{tr } e^{-\beta \tilde{H}}. \end{cases}$$

So we get the following formula

$$(22) \quad Z(\beta) = \text{tr } e^{-\beta \tilde{H}}$$

where β is the Laplace transform variable and it does not necessitate to be interpreted as the "inverse temperature", i.e., $\beta = \frac{1}{\kappa_B \theta}$, where κ_B is the Boltzmann's

constant. If $\beta = \frac{1}{\kappa_B \theta}$ then the system encoded by the observed solution of (\widehat{YM}) , is in equilibrium with a heat bath (canonical system). Note that all above objects are local functions on the space-time N . The same holds for β . We can interpret $Z(\beta)$ as a normalization factor for the local probability density

$$(23) \quad P(E) = \frac{1}{Z} N(E) e^{-\beta E}$$

that the system, encoded by the observed solution, should assume the local energy E , with degeneration $N(E)$. In fact we have:

$$(24) \quad 1 = \int_{Sp(\tilde{H})} P(E) dE = \frac{1}{Z} \int_{Sp(\tilde{H})} N(E) e^{-\beta E} dE = \frac{Z}{Z}.$$

As a by-product we get that the *local average energy* $\langle E \rangle \equiv e$ can be written, by means of the partition function, in the following way:

$$(25) \quad e = -(\partial \beta \ln Z).$$

In fact, one has

$$(26) \quad \left\{ \begin{array}{l} e = \int_{Sp(\tilde{H})} E P(E) dE = \frac{1}{Z} \int_{Sp(\tilde{H})} E N(E) e^{-\beta E} dE \\ = \frac{1}{Z} \int_{Sp(\tilde{H})} E \text{tr} \delta(E - \tilde{H}) e^{-\beta E} dE \\ = \frac{1}{Z} \text{tr} (\tilde{H} e^{-\beta \tilde{H}}) = -\frac{1}{Z} (\partial \beta \cdot Z) = -(\partial \beta \cdot \ln Z). \end{array} \right.$$

When we can interpret $\beta = \frac{1}{\kappa_B \theta}$, then one can write

$$(27) \quad e = \kappa_B \theta^2 (\partial \theta \cdot \ln Z).$$

Then we get also that the *local energy fluctuation* is expressed by means of the variance of e :

$$(28) \quad \langle (\Delta E)^2 \rangle \equiv \langle (E - e)^2 \rangle = (\partial \beta \partial \beta \cdot \ln Z).$$

Furthermore, we get for the *local heat capacity* C_v the following formula:

$$(29) \quad C_v = (\partial \theta \cdot e) = \frac{1}{\kappa_B \theta^2} \langle (\Delta E)^2 \rangle.$$

We can define the *local entropy* by means of the following formula:

$$(30) \quad s = -\kappa_B \int_{Sp(\tilde{H})} P(E) \ln P(E) dE.$$

In fact one can prove that one has the usual relation by means of the energy. Really we get:

$$(31) \quad \left\{ \begin{array}{l} s = -\kappa_B \int_{Sp(\tilde{H})} P(E) \ln P(E) dE \\ = \kappa_B (\ln Z + \beta e) = (\partial \theta \cdot (\kappa_B \theta \ln Z)). \end{array} \right.$$

Then from the relation $s = \kappa_B (\ln Z + \beta e)$ we get $e = \theta s - \kappa_B \ln Z$, hence also $(\partial s \cdot e) = \theta$. This justifies the definition of entropy given in (30). Furthermore, from (31) we get also $\frac{1}{\beta} \ln Z = e - \theta s = f$, where f is the Helmholtz free energy. It follows the following expression of the local Helmholtz free energy, by means of the local partition function Z :

$$(32) \quad f \equiv e - \theta s = -\kappa_B \theta \ln Z.$$

Conversely, from (32) it follows that the partition function can be expressed by means of the local Helmholtz free energy

$$(33) \quad Z = e^{-\beta f}.$$

So we see that the local thermodynamic functions, can be expressed as scalar-valued differential operators on the fiber bundle $W[i] \times_N T_0^0 N \rightarrow N$.

The concept of quantum states can be also related to a proof for existence of solutions with mass-gap. In fact, we have proved that equation (YM) admits local and global solutions with mass-gap. These are contained into a sub-equation, $(Higgs\text{-}quantum\text{ super PDE})$, $(Higgs) \subset (YM)$, that is formally integrable and completely integrable, and also a stable quantum super PDE. If $H_3(M; \mathbb{K}) = 0$, $(Higgs)$ is also a quantum extended crystal super PDE. In general solutions contained in $(Higgs)$ are not stable in finite times. However there exists an associated stabilized quantum super PDE, (resp. quantum extended crystal super PDE), where all global smooth solutions are stable in finite times. Furthermore, there exists a quantum super partial differential relation, $(quantum\ Goldstone\text{-}boundary)$, $(Goldstone) \subset (YM)$, bounding $(Higgs)$, such that any global solution of (YM) , loses/acquires mass, by crossing $(Goldstone)$.

Example 4.2. Now, let us answer to the question: "Do pictures in Fig. 4 represent possible smooth integral manifolds, i.e., solutions, of (YM) , or $(YM)[i]$, with respect to a quantum relativistic frame i ?"

Taking into account above results on (YM) and $(YM)[i]$, we can answer, "yes", accepted that the initial and final Cauchy data have the same quantum numbers, identified by the quantum conservation laws of (YM) and $(YM)[i]$ respectively. Furthermore, we can state that such solutions must necessarily be singular ones, since in order that should be smooth, it is necessary that V should be diffeomorphic to $N_0 \times \hat{D}^{1|1}$ or $N_0 \times I$. Then in such a case we should also have $N_0 \cong N_1$. Therefore we can conclude that quantum nonlinear propagators representing reactions considered in Example 2.11 cannot be smooth solutions, but singular ones.

Lemma 4.3. The quantum mass of a solution of (YM) can be written, in coordinates adapted to a quantum relativistic frame, in the form reported in (34).

$$(34) \quad m = \frac{1}{2}|\tilde{S}|^2 - \frac{1}{2}\hat{\Delta} + m_{\odot} + m_{\mathfrak{A}}.$$

Proof. The quantum Hamiltonian can be written in the form reported in (35).

$$(35) \quad H = \mathbb{R}H + \odot H + \mathfrak{A}H.$$

This follows directly, by considering the expression for the quantum Hamiltonian, by using the splitting induced by the quantum superalgebra \mathfrak{g} , and by taking into account the condition that the solution admits a spin structure. The corresponding calculus is reported in (36) and following ones.

$$(36) \quad \left\{ \begin{array}{l} H = \frac{1}{2}(\tilde{R}_{\alpha\beta}^K \tilde{R}_K^{\alpha\beta} - \tilde{\mu}_{\alpha\beta}^K \tilde{R}_K^{\alpha\beta}) \\ = \frac{1}{2}(\mathbb{R}\tilde{R}_K^{\alpha\beta} \mathbb{R}\tilde{R}_K^{\alpha\beta} - \mathbb{R}\tilde{\mu}_{\alpha\beta}^K \mathbb{R}\tilde{R}_K^{\alpha\beta}) \\ + \frac{1}{2}(\odot\tilde{R}_K^{\alpha\beta} \odot\tilde{R}_K^{\alpha\beta} - \odot\tilde{\mu}_{\alpha\beta}^K \odot\tilde{R}_K^{\alpha\beta}) \\ + \frac{1}{2}(\mathfrak{A}\tilde{R}_K^{\alpha\beta} \mathfrak{A}\tilde{R}_K^{\alpha\beta} - \mathfrak{A}\tilde{\mu}_{\alpha\beta}^K \mathfrak{A}\tilde{R}_K^{\alpha\beta}) \\ = \mathbb{R}H + \odot H + \mathfrak{A}H. \end{array} \right.$$

In coordinates adapted to the quantum relativistic frame, we get that $\psi^\lambda = \delta_0^\lambda$, and since

$$\mathbb{R}\tilde{\mu}_K^\lambda \mathbb{R}\tilde{R}_{\alpha\beta}^K = \tilde{S}_{\alpha\beta}^\lambda = \tilde{s}_{\alpha\beta} \psi^\lambda$$

we get $\mathbb{R}\tilde{\mu}_K^i \mathbb{R}\tilde{R}_{\alpha\beta}^K = \tilde{S}_{\alpha\beta}^i = 0$ and $\mathbb{R}\tilde{\mu}_K^0 \mathbb{R}\tilde{R}_{\alpha\beta}^K = \tilde{S}_{\alpha\beta}^0 = \tilde{s}_{\alpha\beta}$. Therefore we have

$$(37) \quad \left\{ \begin{array}{l} |\tilde{S}|^2 = |\tilde{s}|^2 = \mathbb{R}\tilde{\mu}_K^\lambda \mathbb{R}\tilde{R}_{\alpha\beta}^K \mathbb{R}\tilde{R}_K^{\alpha\beta} \mathbb{R}\tilde{\mu}_\lambda^{\tilde{K}} \\ = \mathbb{R}\tilde{\mu}_K^0 \mathbb{R}\tilde{\mu}_0^{\tilde{K}} \mathbb{R}\tilde{R}_{\alpha\beta}^K \mathbb{R}\tilde{R}_K^{\alpha\beta} + \hat{\Delta} \\ = \delta_K^{\tilde{K}} \mathbb{R}\tilde{R}_{\alpha\beta}^K \mathbb{R}\tilde{R}_K^{\alpha\beta} + \hat{\Delta} \\ = \mathbb{R}\tilde{R}_{\alpha\beta}^K \mathbb{R}\tilde{R}_K^{\alpha\beta} + \hat{\Delta} \end{array} \right.$$

with

$$(38) \quad \hat{\Delta} \equiv \mathbb{R}\tilde{\mu}_K^0 [\mathbb{R}\tilde{R}_{\alpha\beta}^K, \mathbb{R}\tilde{\mu}_0^{\tilde{K}}] \mathbb{R}\tilde{R}_K^{\alpha\beta} + \mathbb{R}\tilde{\mu}_K^0 \mathbb{R}\tilde{R}_{\alpha\beta}^K [\mathbb{R}\tilde{R}_K^{\alpha\beta}, \mathbb{R}\tilde{\mu}_0^{\tilde{K}}].$$

So, in general, we can write

$$(39) \quad \mathbb{R}\tilde{R}_{\alpha\beta}^K \mathbb{R}\tilde{R}_K^{\alpha\beta} = |\tilde{S}|^2 - \hat{\Delta}$$

and by using equation (36) we get the expression (40) for the quantum Hamiltonian in terms of quantum torsion.

$$(40) \quad H = \frac{1}{2}|\tilde{S}|^2 - \frac{1}{2}\hat{\Delta} + \odot H + \star H.$$

As a by-product, we get that the quantum mass of an observed solution by means of a quantum relativistic frame $i : N \rightarrow M$, of $(YM)[i]$, the alternative representation (34) in terms of quantum torsion. \square

Definition 4.4. *Let us define reduced quantum mass of an observed solution by means of a quantum relativistic frame $i : N \rightarrow M$, of $(YM)[i]$:*

$$M \equiv m + \frac{1}{2}\hat{\Delta} - m_\odot - m_\star.$$

Theorem 4.5. *There is the following direct relation between reduced quantum mass and quantum torsion:*

$$M = \frac{1}{2}|\tilde{S}|^2.$$

Definition 4.6. *Let us define phenomenological quantum spin of an observed solution by means of a quantum relativistic frame $i : N \rightarrow M$, of $(YM)[i]$, a \hat{A} -valued quantum scalar J such that:²²*

$$\sqrt{J} \equiv |\tilde{S}|^2.$$

Corollary 4.7 (Quantum Regge-type trajectories). *There is the following direct relation between reduced quantum mass and phenomenological quantum spin:*

$$(41) \quad M^2 = \frac{1}{4}J.$$

We call quantum Regge-type trajectories the relation (41) between quantum square mass M^2 and the phenomenological quantum spin J .

• Another question, in a sense related to the previous problem, is the dynamical justification of the Gell-Mann-Nishijima formula. This actually has not yet found

²²The natural meaning of \sqrt{J} is the \hat{A} -valued function on N , such that: $\sqrt{J}\sqrt{J} = J$.

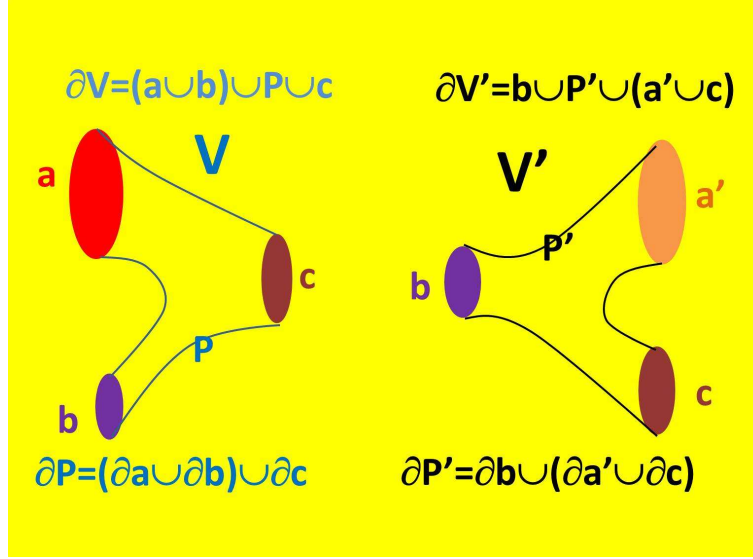


FIG. 6. Representation of a quantum nonlinear propagator V , for a reaction $a + b \rightarrow c$, and its quantum nonlinear anti-propagator V' , corresponding to the reaction $b \rightarrow a' + c$, obtained for quantum crossing symmetry.

a dynamical derivation.²³ Really dynamical characterizations of hypercharge and 3th isospin component, can be obtained by means of electric charge, by giving the dynamical expression of this last quantity with respect to a quantum relativistic frame. This is obtained in the following theorem.

Theorem 4.8 (Quantum Gell-Mann-Nishijima formula and electric-charge-gap-solutions). *We define square-fundamental electric charge of an observed solution V of $(YM)[i]$, the littlest value in the spectrum $Sp(\hat{w})$ of the observed quantum electromagnetic energy \hat{w} , and denote it by q^2 . We say that V has an electric-charge gap if $q^2 > 0$. (In general $q^2 \geq 0$.) The quantum electric charge, $\hat{Q}(t)$, of an observed solution V of $(YM)[i]$, at the proper time t , is expressed by the formula (42).*

$$(42) \quad \hat{Q}^2(t) = \int_{\sigma_t} \left[\odot \tilde{R}_{0i}^K \odot \tilde{R}_{0K}^i + \star \tilde{R}_{0i}^K \star \tilde{R}_{0K}^i + \oplus \tilde{B}_{0i}^K \oplus \tilde{B}_{0K}^i + \odot \tilde{B}_{0i}^K \odot \tilde{B}_{0K}^i + \star \tilde{B}_{0i}^K \star \tilde{B}_{0K}^i \right] \otimes \eta \in A$$

where $\eta = \sqrt{g_{ij}} d\xi^1 \wedge d\xi^2 \wedge d\xi^3$ is the canonical space-like volume form on $\sigma_t \subset V$. This last is the 3-dimensional space-like sub-manifold of V at the time t . Furthermore, we call quantum hypercharge, $\hat{Y}(t)$, (resp. quantum 3th-isospin component, $\hat{I}_3(t)$), at the proper time t , the elements of the quantum (super)algebra A , such that holds the formula (43) (quantum Gell-Mann-Nishijima formula).

$$(43) \quad \hat{Q}^2(t) = (\hat{I}_3(t) + \frac{1}{2}\hat{Y}(t))^2, \forall t \in \Delta$$

where Δ is the definition time-set of the considered solution of $(YM)[i]$.

²³The more interesting effort in this direction was an heuristic semiclassical justification given in [15].

- The spectral content of $\hat{Q}(t)$ is given by $Sp(\pm\sqrt{\hat{Q}(t)^2})$.
- There exists an open sub-equation $(\widehat{YM})[i]_w \subset (\widehat{YM})[i]$, quantum electromagnetic-Higgs PDE, that is formally integrable and completely integrable, where live all solutions with electric-charge gap. The boundary $\partial(\widehat{YM})[i]_w = \overline{(\widehat{YM})[i]_w} \setminus (\widehat{YM})[i]_w \subset (\widehat{YM})[i]$, is a partial differential relation that we call quantum electromagnetic-Goldstone boundary and denote by $(\widehat{Goldstone})[i]_w$. An electrically neutral connected, simply connected, 3-dimensional Cauchy data $N \subset (\widehat{YM})[i]$, cannot be contained into $(\widehat{YM})[i]_w$. Let V be a quantum nonlinear propagator, such that $\partial V = N_0 \sqcup P \sqcup N_1$, with $N_0 \subset (\widehat{YM})[i]_w$ and $N_1 \not\subset (\widehat{YM})[i]_w$. Let us assume that N_r , $r = 0, 1$, are connected, simply connected particles, hence N_0 has an electric-charge gap. The particle N_1 , instead cannot have electric-charge gap. Thus, V , by crossing $(\widehat{Goldstone})[i]_w$, loses its electric-charge gap, passing from N_0 to N_1 , and vice versa.

Proof. The solution V identifies a quantum-electric-charge field $\hat{E} = \psi] \tilde{R} = \textcircled{\mathbb{R}} \hat{E} + \textcircled{\mathbb{C}} \hat{E} + \textcircled{\mathbb{A}} \hat{E}$ and a quantum-magnetic-charge field $\hat{B} = \psi](\star \tilde{R}) = \textcircled{\mathbb{B}} \hat{B} + \textcircled{\mathbb{C}} \hat{B} + \textcircled{\mathbb{A}} \hat{B}$. Assuming that the solution V has a quantum spin, one has $\textcircled{\mathbb{R}} \hat{E} = 0$. (See [82].) Then the quantum electro-magnetic-charge energy of the solution of $(\widehat{YM})[i]$ is given by equation (44).

$$(44) \quad \hat{w} = \hat{w}_e + \hat{w}_m = \textcircled{\mathbb{C}} \tilde{R}_{0i}^K \textcircled{\mathbb{C}} \tilde{R}_{0K}^i + \textcircled{\mathbb{A}} \tilde{R}_{0i}^K \textcircled{\mathbb{A}} \tilde{R}_{0K}^i + \textcircled{\mathbb{B}} \tilde{B}_{0i}^K \textcircled{\mathbb{B}} \tilde{B}_{0K}^i + \textcircled{\mathbb{C}} \tilde{B}_{0i}^K \textcircled{\mathbb{C}} \tilde{B}_{0K}^i + \textcircled{\mathbb{A}} \tilde{B}_{0i}^K \textcircled{\mathbb{A}} \tilde{B}_{0K}^i.$$

Therefore the quantum electromagnetic energy of the space-like set σ_t is given by the expression on the right in (42). We define *quantum electric charge* contained, at the proper time t , into a space-like set $\sigma_t \subset V$, $\hat{Q}(t) \in A$, such that $\hat{Q}^2(t) = \int_{\sigma_t} \hat{w} \otimes \eta$.²⁴ Then equation (43) gives a dynamical definition for \hat{Y} and \hat{I}_3 .

To prove the last part of the theorem it is enough to consider the continuous mapping $\hat{w} : (\widehat{YM})[i] \rightarrow \hat{A}$, defined by means of equation (44). Set $(\widehat{YM})[i]_w \equiv (\hat{w})^{-1}(G(\hat{A})) \subset (\widehat{YM})[i]$. Then $(\widehat{YM})[i]_w$ is an open quantum PDE of $(\widehat{YM})[i]$, hence retains the same formal properties of this last equation. The rest of the proof follows the same line of the ones of Theorem 3.28 in [82] about solutions with mass-gap. Let us only emphasize that a neutral particles $N_0 \subset (\widehat{YM})[i]$, with trivial topology, i.e., connected and simply connected, cannot admit $0 \in Sp(\hat{Q}(t))$, whether $N_0 \subset (\widehat{YM})[i]_w$, since in $(\widehat{YM})[i]_w$, one has $\lambda > 0$, for any $\lambda \in Sp(\hat{w})$. Furthermore, if the quantum nonlinear propagator V is such $\partial V = N_0 \sqcup P \sqcup N_1$, assuming that $N_1 \not\subset (\widehat{YM})[i]_w$ and that it has trivial topology, then N_1 cannot have electric-charge gap, hence V crossing $(\widehat{Goldstone})[i]_w$ must necessarily lose electric-charge gap. \square

Remark 4.9. *Theorem 4.8 does not necessarily contradict the conservation of the quantum electric charge. In fact, we can have two Cauchy data $a, b \subset (\widehat{YM})[i]_w$*

²⁴This is justified taking into account that the electric charge contained into a 3-dimensional set having the electromagnetic energy W , is $q^2 = 2C W$, with C the capacity of this set. Therefore $Q^2 = \frac{q^2}{2C}$. i.e., the formula (42) is normalized with respect to a factor form $\kappa_{\text{electric-charge}}(t) = \frac{1}{2C(t)}$.

and another one $c \notin \widehat{(YM)}[i]_w$, such that $\langle \hat{q}, a \rangle = -\langle \hat{q}, b \rangle \neq 0$, where $\langle \hat{q}, a \rangle = \sqrt{\int_a \hat{f} \otimes \eta} = -\sqrt{\int_b \hat{f} \otimes \eta}$, with \hat{f} the function under integral in (42). Furthermore, let $\langle \hat{q}, c \rangle = 0$. Then in the integral bordism classes $[a \sqcup b = N_0]$ and $[N_1]$, one has $\langle \hat{q}, [a \sqcup b] \rangle = 0 = \langle \hat{q}, c \rangle$. Therefore, for a quantum nonlinear propagator V , such that $\partial V = N_0 \sqcup P \sqcup N_1$ of $\widehat{(YM)}[i]$ one has $\langle \alpha, a \rangle + \langle \alpha, b \rangle = -\langle \alpha, P \rangle$, for $\alpha = \hat{q}$, since $\langle \hat{q}, P \rangle = 0$. This is surely down since P is a time-like manifold having 2-dimensional space-like sections σ_t , hence $\langle \hat{q}, P \rangle = \sqrt{\int_{\sigma_t} \hat{f} \otimes \sqrt{g_{ij}} d\xi^1 \wedge d\xi^2 \wedge d\xi^3} = 0$.

From Theorem 4.8 and Remarks 4.9 it follows the following important theorem.

Theorem 4.10 (*Q-exotic quantum nonlinear propagators of $\widehat{(YM)}[i]$*). *For any observed quantum nonlinear propagator V of $\widehat{(YM)}[i]$, such that $\partial V = N_0 \sqcup P \sqcup N_1$, where N_i , $i = 0, 1$, are 3-dimensional space-like admissible Cauchy data of $\widehat{(YM)}[i]$, and P is a suitable time-like 3-dimensional integral manifold with $\partial P = \partial N_0 \sqcup \partial N_1$, equation (45) holds.*

$$(45) \quad \hat{Q}[i|t_0] = \hat{Q}[i|t_1] \bmod \Omega[V] \in A$$

where $\hat{Q}[i|t_r] \in A$ is the quantum electric charge on N_r , $r = 0, 1$, and $\Omega[V] \in A$, is a term that in general is not zero and that we call lost quantum electric-charge. We call Q-exotic quantum nonlinear propagators, quantum nonlinear propagators such that $\Omega[V] \neq 0 \in A$.²⁵

Proof. The proof follows the same strategy of one considered to prove Theorem 3.20 in part I [88]. In fact the gauge invariance of equation $\widehat{(YM)}[i]$ produces a quantum integral characteristic 3-form, in the sense of Lemma 3.19 in part I, that has a structure similar to ω_H . Let us denote such a conservation law by

$$\omega_q = (\omega_q)_0 \otimes d\xi^1 \wedge d\xi^2 \wedge d\xi^3 + \sum_{1 \leq i \leq 3} (\omega_q)_i \otimes d\xi^0 \wedge d\xi^1 \wedge \dots \wedge \widetilde{d\xi^i} \wedge d\xi^3,$$

with $\widetilde{d\xi^i}$, absent, $i = 1, 2, 3$, and $(\omega_q)_\alpha(p) \in A$, for $p \in V$. The quantum electric charge, on a space-like section σ_t of V , is given by means of ω_q , by the following equation

$$\hat{Q}[i|t] = \int_{\sigma_t} (\omega_q)_0|_{\sigma_t} \otimes d\xi^1 \wedge d\xi^2 \wedge d\xi^3.$$

Therefore we have

$$\hat{Q}[i|t_0] - \hat{Q}[i|t_1] = \int_P (\omega_q)_i \otimes d\xi^0 \wedge d\xi^1 \wedge \dots \wedge \widetilde{d\xi^i} \wedge \dots \wedge d\xi^3 \equiv \Omega[V].$$

In general $\Omega[V] \neq 0$, hence we get $\hat{Q}[i|t_0] = \hat{Q}[i|t_1] \bmod \Omega[V]$. We call $\Omega[V] \in A$ *lost quantum electric-charge* of the observed quantum nonlinear propagator V of $\widehat{(YM)}[i]$. In other words, in general one has $\frac{d}{dt}\hat{Q}(t) \neq 0 \in A$. \square

Remark 4.11. *In the Part III we will further characterize Q-exotic quantum nonlinear propagators as nonlinear effects of exotic quantum supergravity.*

²⁵This agrees with the conservation of the observed quantum Hamiltonian. See Theorem 3.20 in [88]. In fact, the observed quantum electromagnetic energy is a form of quantum energy, even if, in general, it does not coincide with the observed quantum Hamiltonian.

- In the following we shall prove that the phenomenological crossing symmetry in particle reactions is well justified in the Prástaro's algebraic topology theory of quantum super PDE.

Theorem 4.12 (Quantum crossing symmetry). *If (\widehat{YM}) admits a quantum nonlinear propagator V such that $\partial V = N_0 \sqcup P \sqcup N_1$, with $N_0 = a \sqcup b$ and $N_1 = c \sqcup d \sqcup e$, then there exists also a quantum nonlinear propagator V' such that $\partial V' = N'_0 \sqcup P' \sqcup N'_1$, with $N'_1 = a' \sqcup b'$ and $N'_0 = c' \sqcup d' \sqcup e'$, where the primed symbols denote antiparticles. This property is called quantum crossing symmetry. Similarly there exist quantum nonlinear propagators between $a \sqcup b \sqcup c'$ and $d \sqcup e$ or between b and $a' \sqcup c \sqcup d \sqcup e$.²⁶*

Proof. This follows from the general relation between quantum integral bordism groups and quantum integral characteristic conservation laws. (See [74].) In fact if there exists a quantum nonlinear propagator V such that $\partial V = N_0 \sqcup P \sqcup N_1$, with $N_0 = a \sqcup b$ and $N_1 = c \sqcup d \sqcup e$, then this means that $N_1 \in [N_0 \sqcup P] \in \text{Bor}_{3|3}^{(\widehat{YM})}$, hence $N_0 \sqcup P \sqcup N_1 \in [0] \in \text{Bor}_{3|3}^{(\widehat{YM})}$. Since antiparticles reverse quantum integral characteristic numbers, it follows that for any quantum integral characteristic conservation law α of (\widehat{YM}) , one has $\langle [\alpha], [N_0 \sqcup P \sqcup N_1] \rangle = 0 = \langle [\alpha], [N'_1 \sqcup P' \sqcup N'_0] \rangle$. Therefore, $(N'_1 \sqcup P') \in [N'_0] \in \text{Bor}_{3|3}^{(\widehat{YM})}$, hence there exists a quantum nonlinear propagator V' such that $\partial V' = N'_0 \sqcup P' \sqcup N'_1$. Note also, that as by-product, we get that there exist a $(3|3)$ -dimensional quantum integral supermanifold P' , such that $\partial P' = \partial N'_1 \sqcup \partial N'_0$, similarly to what happens for P , namely $\partial P = \partial N_0 \sqcup N_1$. This means that one has $\partial N'_1 \in [\partial N'_0] \in \Omega_{2|2}^{(\widehat{YM})}$ and $\partial N_1 \in [\partial N_0] \in \Omega_{2|2}^{(\widehat{YM})}$. Similarly one proves existence of the other reactions obtained from crossing symmetry. For example to the quantum nonlinear propagator V , such that $\partial V = (a \sqcup b) \sqcup P \sqcup c$, corresponds for quantum crossing symmetry, the following quantum nonlinear propagator V' , such that $\partial V' = b \sqcup P' \sqcup (a' \sqcup c)$, where P' is the anti- P . (See Fig. 6 for a representation of P and its anti- P .) There are cases where can exist more than only one anti- P . □

- The following theorem answers to the question: "Do exist massive photons?".

Theorem 4.13 (Existence of quantum massive photons). *The quantum super Yang-Mills equation (\widehat{YM}) admits solutions that starting from a Cauchy data $N_0 = a \sqcup b$, where a represents a quantum electron and b a quantum positron, bords N_1 , representing a quantum massive particle. Any annihilation $e^+ + e^- \rightarrow \gamma + \gamma$, must necessarily generate a quantum virtual massive photon, say γ_m , before to produce a couple of massless photons γ . Conversely a quantum massive photon decays giving $\gamma_m \rightarrow e^+ + e^-$.*

²⁶The crossing symmetry is a well-known property in the reaction particles phenomenology. These, besides a set of other symmetries and phenomenological conservation laws constitute the Holy Bible for particle-reaction physicists. (See Tab. 7.) In Tab. 7 are reported also two quantum reactions related by means of crossing symmetry. These are the *neutron decay* and *neutrino detection*.)

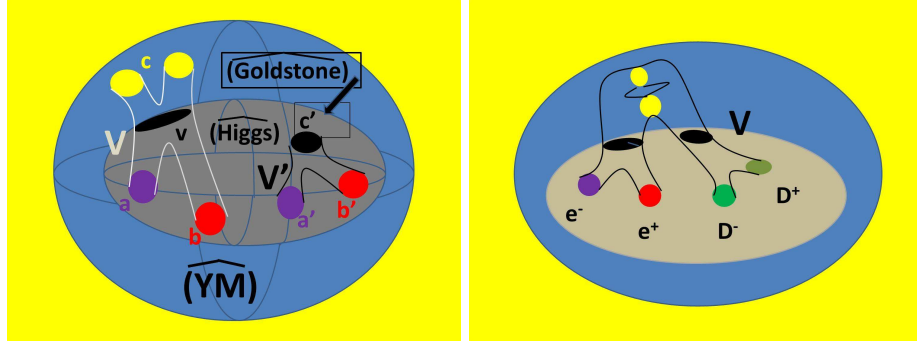


FIG. 7. In the figure on the left it is represented a couple of zero-mass photons ($c \notin (\widehat{Higgs})$) and massive photons ($c' \in (\widehat{Higgs})$). The quantum non-linear propagator V bordering a couple of massless photons (yellow particles) with electron-positron ($a \sqcup b$) has a *Goldstone piece* (black quantum virtual particle). This is identified with a massive photon. In other words the annihilation of electron-positron generates a virtual massive photon before to produce a couple of massless photons. The quantum non-linear propagator V' bordering the massive photon (black) with electron-positron ($a' \sqcup b'$) has not a Goldstone piece, since it is completely inside $(\widehat{Higgs}) \subset (\widehat{YM})$. This is a massive photon decay. In the figure on the right, it is represented a quantum nonlinear propagator, V , bordering massive couple of antiparticles, and having a part outside (\widehat{Higgs}) (the grey region). In this case we can talk of two Goldstone pieces in V , i.e., there are two massive photons (black particles) related by V .

Proof. Let us first identify a quantum massive photon with a quantum massive particle γ_m with decay into an electron-positron couple. We know that from Theorem 3.28 in [82] can exist a quantum propagator V' bordering $a' \sqcup b'$ with c' , where $a' \sqcup b'$ is a couple electron-positron and c' is a massive particle, hence all contained in $(\widehat{Higgs}) \subset (\widehat{YM})$. On the other hand, from the standard Compton scattering $\gamma + e^- \rightarrow e^- + \gamma$ we say, for crossing symmetry, that holds the following reaction (*annihilation electron-positron*): $e^+ + e^- \rightarrow \gamma + \gamma$. Theorem 3.28 in [82] assures the existence of a quantum nonlinear propagator V such that $\partial V = N_0 \sqcup P \sqcup N_1$, where N_0 , is contained in the sub-equation $(\widehat{Higgs}) \subset (\widehat{YM})$, and $N_1 \notin (\widehat{Higgs})$. Therefore V , should cross the Goldstone boundary $(\widehat{Goldstone}) \subset (\widehat{YM})$, hence V should contain a Goldstone piece. This proves that the annihilation electron-positron implies to pass across a quantum massive photon: (Note that whether it is permitted the reaction $e^+ + e^- \rightarrow \gamma_m$, for *crossing symmetry* it is also permitted the decay $\gamma_m \rightarrow e^- + e^+$.)²⁷ In Fig. 7 are represented productions of quantum massless

²⁷Actually there are attempts to give experimental evidence to the existence of massive photons. (Visit (Thomas Jefferson National Accelerator Facility, Newport News, Virginia, USA), and [5].) Massive photons can be identified with neutral massive bosonic particles. In the particle-zoo these could be identified with so-called *vector bosons*, like neutral ρ -meson and φ -meson and Z^0 -boson, all having spin 1.

photons and quantum massive photons with respect to the constraints (\widehat{Higgs}) and $(\widehat{Goldstone})$ in (\widehat{YM}) . Whether e^+ and e^- collide at high energy, their annihilation can be reversed into a massive couple of mesons $D^+ \sqcup D^-$. In other words there exists a quantum nonlinear propagator V , bording $e^+ \sqcup e^-$ with $D^+ \sqcup D^-$, such that $\partial V = (e^+ \sqcup e^-) \cup P \cup (D^+ \sqcup D^-)$, where P is an integral quantum supermanifold, partially outside (\widehat{Higgs}) , but yet contained into (\widehat{YM}) . (See Fig. 7.) \square

From above considerations, we see that the concept of massive photons can be generalized to more large set of particles.

Definition 4.14 (Quantum anti-particles and quantum massive photons). *We say that two quantum, massive, electric charged particles $a \sqcup a' \subset (\widehat{YM})$ are quantum anti-particles, if they have the same mass and opposite quantum numbers.*

We call a-quantum massive photon a quantum massive uncharged particle, having a decay into a couple (a, a') of quantum massive, electric charged anti-particles. We denote such a quantum massive photon with the symbol γ_m^a , or simply γ_m , if no confusion can arise. Therefore, we can write $\gamma_m^a \rightarrow a + a'$.

Theorem 4.15 (Existence of a-quantum massive photons). *For any couple (a, a') of quantum massive, electric charged anti-particles, there exists a a-quantum massive photon γ_m^a .*

Proof. The proof can be obtained by analogy of the one for Theorem 4.13. \square

Example 4.16 (Existence of p -quantum massive photon). *Since it is permitted the Compton scattering $p + n\gamma \rightarrow p + n\gamma$, for suitable $n \in \mathbb{N}$, it is also permitted for quantum crossing symmetry the following one $p + \bar{p} \rightarrow 2n\gamma$. Therefore, a virtual p -quantum massive photon γ_m^p necessarily exists.²⁸*

Definition 4.17 (Quantum massive neutrinos). *We say that a quantum, massive, quasi-particle with zero electric charge is a quantum massive neutrino, if it admits a decay into a couple (neutrino, anti-neutrino) of the same type.*

Theorem 4.18 (Existence of quantum massive neutrinos). *The quantum super Yang-Mills equation (\widehat{YM}) admits solutions that represent decays of quantum massive neutrinos.*

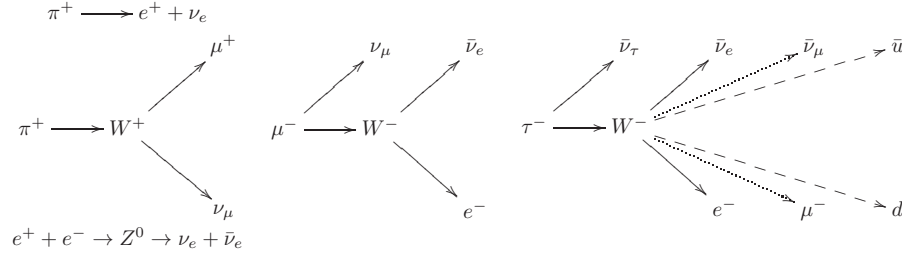
Proof. In fact, we can find a quantum nonlinear propagator $V \subset (\widehat{YM})$, such that $\partial V = N_0 \cup P \cup N_1$, where $N_0 = a \sqcup b \subset (\widehat{YM})$, with $(a, b) = (\nu_e, \bar{\nu}_e) \notin (\widehat{Higgs})$, and $N_1 = c \sqcup d \subset (\widehat{Higgs})$, with (c, d) ia couple of antiparticles in the sense of Definition 4.14. Then, necessarily V must cross $(\widehat{Goldstone})$, hence identifies a

²⁸From some experimental results it is well-known that the annihilation $p + \bar{p}$ produces photons γ , through intermediate reactions and decays coming from massive particles. (For example: $p + \bar{p} \rightarrow 3\pi^0$, $p + \bar{p} \rightarrow 2\pi^0 + \eta$, $p + \bar{p} \rightarrow \pi^0 + 2\eta$, $\pi^0 \rightarrow 2\gamma$, $\eta \rightarrow 2\gamma$, $Z^0 \rightarrow e^+ + e^- \rightarrow 2\gamma$.) Therefore, the virtual p -quantum massive photon γ_m^p has a very complex structure, made by a collection of massive particles, like mesons π and η , bosons like Z^0 and leptons e^\pm , all inside (\widehat{Higgs}) . Note that in the virtual massive photon γ_m^p enters also the vector boson Z^0 that is more massive than proton. ($m_p = 0.938 \text{ GeV}/c^2$, $m_Z = 91 \text{ GeV}/c^2$ and $m_W = 80 \text{ GeV}/c^2$.) This means that virtual massive photons can have very large masses. (See Remark 4.23.) Massive intermediate vector bosons W^\pm and Z^0 , were predicted from Steven Weinberg, Abdus Salam and Sheldon Glashow in 1979 (Nobel award 1979) and experimental discovered by Carlo Rubbia in 1983 (Nobel award 1984).

massive quasi-particle, say $\nu_m \subset (\widehat{Higgs})$. Then, applying the quantum crossing symmetry to the reaction $\nu_e + \bar{\nu}_e \rightarrow \nu_m \rightarrow c + d$, we get also the reaction $d + c \rightarrow \nu_m \rightarrow \bar{\nu}_e + \nu_e$, i.e., there exists a quantum nonlinear propagator of (\widehat{YM}) that encodes such a reaction. This proves that the massive, neutral particle ν_m , decays into the couple $(\nu_e, \bar{\nu}_e)$, therefore ν_m is a massive neutrino.

Let us emphasize that neutrinos are of three different type: ν_e, ν_μ and ν_τ , called e -neutrino, μ -neutrino and τ -neutrino respectively. They differ for the type of decay that produce them. For example see (46).

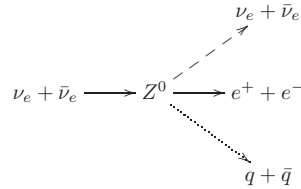
(46)



In 46) the dot-arrows and dash-arrows starting from W^- are alternative to the full-arrows and between them. In all these cases there is an intermediate production of a quasi particle Z^0 or W^\pm . Such quasi-particles can be usually produced when a massive solution cross $(\widehat{Goldstone})$. Therefore from this point of view we should have a further argument to consider the usual neutrinos massless particles, according to the Gell-Mann's standard model. However, the quasi-particles W_\pm cannot be considered massive neutrinos, since they do not decay into a massless couple (neutrino, anti-neutrino). Instead Z^0 has all the properties to be considered a massive neutrino ! \square

Example 4.19 (What dark matter is ?). *Theorem 4.18 agrees with the experimental fact that the annihilation of the couple $(\nu_e, \bar{\nu}_e)$ produces a Z^0 massive boson. This last can again decay in the couple the couple $(\nu_e, \bar{\nu}_e)$, or in a couple lepton-antilepton, or quark-antiquark, according to energy level considered. For example:*

(47)



In (47) dot-arrow and dash-arrow starting from Z^0 are alternative to the full-arrow and each other one. Then the Z boson can be considered a massive neutrino when decay into the couple $(\nu_e, \bar{\nu}_e)$. The quantum crossed symmetry reaction $\nu_e + \bar{\nu}_e \rightarrow Z^0$, is an example showing how massless particles can cross $(\widehat{Goldstone})$, producing massive particles inside (\widehat{Higgs}) . The reaction, $e^+ + e^- \rightarrow Z^0 \rightarrow \nu_e + \bar{\nu}_e$, obtained, by quantum cross symmetry, from the reaction in the middle line of (47), shows that the annihilation of the couple (e^-, e^+) , when it is enough energetic, produces a massive quasi-particle Z^0 , before to destroy its mass, crossing $(\widehat{Goldstone})$, and

producing the couple of massless particles $(\nu_e, \bar{\nu}_e)$. This clarifies that the energetic level of an electric-charged, massive couple (particle, antiparticle) decides whether in a scattering will produce a massive photons or a massive neutrino.²⁹

Let us also emphasize that the so-called “dark matter” can be interpreted as massive virtual particles codified in Theorem 4.13, Theorem 4.15 and Theorem 4.18. In other words “dark matter” can be considered a generic term to identify virtual massive particles produced when a solution of the quantum super Yang-Mills equation (YM) , crosses the Goldstone boundary, coming inside the Higgs quantum super PDE, $(Higgs) \subset (YM)$ contained into (YM) . This interpretation for “dark matter” could theoretically support some recent experimental observations where positron fraction is stably increasing at high energy levels, that are just necessary to produce massive photons, massive neutrinos and other a-quantum massive photons, according to the above quoted theorems. See [3] for recent experiments that should suggest existence of “dark matter” in the sense here specified.³⁰

- The following theorem answers to the question: “Do quarks are fundamental particles?”

Theorem 4.20 (Quarks break-down). *Quarks are not fundamental particles.*

Proof. The proof can be obtained easily by considering Theorem 4.13. In fact for any quark q , we can associate an anti-quark \bar{q} , so that (q, \bar{q}) is a couple of massive, electric-charged anti-particles in the sense of Definition 4.14, hence there exist massive q -photons γ_q . In other words there exist quantum nonlinear propagators allowing reactions $q + \bar{q} \rightarrow n \gamma$, $n \in \mathbb{N}$. In these reaction quarks are transformed in particles that do not contain quarks, hence quarks are broken-down ! To similar conclusions one arrives by considering Theorem 4.18. In fact, by considering the quantum reaction $\bar{q} + q \rightarrow Z^0 \rightarrow \bar{\nu}_e + \nu_e$, obtained by means of quantum cross symmetry of the reaction in the bottom in (47), we see that the mass of the couple (q, \bar{q}) completely disappears and appear particles that are not made by quarks. In other words quarks break-down. \square

- In the following we give a precise meaning to the concept of *observed quantum time* and to the so-called *time-energy uncertainty principle*. This is possible thanks to our geometric theory of quantum (super) PDE’s.³¹

²⁹A guess by B. M. Pontecorvo and V. N. Gribov [29, 59], in order to explain the so-called *mystery of the missing solar neutrinos*, assumed that neutrinos have some oscillations that characterize their different types. But such oscillations require massive neutrinos at least for τ -neutrinos and μ -neutrinos. This fact does not agree with the standard model. On the other hand, from Theorem 4.18 we can understand that massive neutrinos can be identified with some energetic levels of the quasi-particle Z^0 , and that non-electron neutrinos can retain their usual property to be massless particles.

³⁰Production of virtual π^+ decaying into positrons and neutrinos, can be obtained by means massive photons identified with virtual states into γ -nucleons interactions. This is a new point of view looking to long-studied π^+ -photoproduction. (For general information on pions photoproduction see, e.g., [18, 20, 21] and references quoted therein.)

³¹The exact interpretation of the time-energy uncertainty principle was remained an open problem, and not even universally accepted, (see, e.g., L. Landau). Really in order to directly apply the Heisenberg uncertainty principle to the (time, energy) couple, it is necessary to define the meaning of the time as a noncommutative variable, with respect to the energy. But the proper time of a quantum relativistic frame is a commutative variable ! (See [79].) (See also some attempts to

Definition 4.21 (Quantum relativistic observed time). *Let $V \subset \widehat{(YM)}$ be a solution of $\widehat{(YM)}$ and let $\widehat{g} : M \rightarrow \text{Hom}_Z(\dot{T}_0^2 M; A)$ be the quantum metric induced by the quantum graviton, identified by the solution V . Then a quantum relativistic frame $\psi \equiv \{i : N \rightarrow M, g\}$, where (N, g) is a pseudo-Riemannian 4-dimensional manifold, endowed with a time-like flow, identifies on N a A -valued metric $i^*\widehat{g} : N \rightarrow A \otimes S_2^0 N$. We define quantum relativistic observed time, between two points $p_1, p_2 \in N$, belonging to a same flow line of ψ , the A -valued quantum length $\hat{t}[p_1, p_2] \in A$, calculated with respect to $i^*\widehat{g}$. (For more details see [79].) We call spectral content of the quantum relativistic observed time the spectrum $Sp(\hat{t}[p_1, p_2])$ of $\hat{t}[p_1, p_2]$.*

Proposition 4.22. *The quantum relativistic observed time cannot, in general, coincide with the proper time of the quantum relativistic frame.*

Proof. In fact the length of the arc considered in above proposition differs whether measured with respect to g or to $i^*\widehat{g}$, even if one has the case $A = \mathbb{R}$! This can be seen by a direct calculus, by using coordinates $\{\xi^\alpha\}_{0 \leq \alpha \leq 3}$, with $\xi^0 = t$ the proper time of the relativistic quantum frame ψ . Then the arc length between the points p_1, p_2 belonging to the same flow line ϕ_{p_1} , passing for p_1 and p_2 , is given by $s[p_1, p_2] = \int_{[p_1, p_2]} ds = \int_{[t_1, t_2]} \sqrt{g_{\alpha\beta} \dot{\xi}^\alpha \dot{\xi}^\beta} dt = \int_{[t_1, t_2]} \sqrt{g_{00}} dt = t_2 - t_1$, assuming $g_{00} = 1$. On the other hand we have $\hat{t}[p_1, p_2] = \int_{[p_1, p_2]} d\hat{s} = \int_{[t_1, t_2]} \sqrt{|\gamma^* \widehat{g}(s)|} dt \in A$, where $\gamma \equiv \phi_{p_1} \circ i$. (For more details see [79].) In the particular case that $A = \mathbb{R}$, we have that $\hat{t}[p_1, p_2] \in \mathbb{R}$, as well as $s[p_1, p_2]$, but $\gamma^* \widehat{g}(s)$ is different from $\phi_{p_1}^* g$. \square

Theorem 4.23 (Quantum virtual anomaly-massive particles). *A quantum virtual massive quasi-particle can have an observed mass, in the reaction where it appears, larger than the observed energy of incoming particles entering into the reaction.³² However this does not really contradict the Einstein's conservation mass-energy equation.*

Proof. The justification of this fact, that should appear contradict the Einstein's equation of mass-energy conservation $E = mc^2$, is justified by the *Heisenberg's uncertainty principle*. (This is, really, not more a "principle", but a "theorem". See, [65].) In fact, adopting notation used in [65], the uncertainty relation for the commutator $[quantum - time, quantum - energy]$, can be written as reported in equation (48).

$$(48) \quad \sigma^2(\hat{H}) \sigma^2(\hat{t}) \geq \frac{1}{4} |< [\hat{H}, \hat{t}] >|^2.$$

Thus in quantum processes, localized in very short "observed quantum-time", as could happen in the case of virtual particles creation, one could have large masses,

solve this problem by Dirac [17] and L. I. Mandelshtam and I. E. Tamm [44].) Actually the time-energy uncertainty principle is justified by means adopting the Mandelshtam-Tamm's point of view. But that approach is not satisfactory, since it refers to a generic auxiliary non-commutative variable, say B , that has nothing to do with energy and time. Furthermore, in that description time remains a commutative variable, and one identifies the velocity $\frac{dB}{dt}$ with a finite ratio, i.e., $\frac{dB}{dt} \equiv \frac{\Delta B}{\Delta t}$. In fact one uses the following modified uncertainty relation: $\Delta B \frac{\Delta E}{\Delta t} \geq \hbar$. Then by using the approximation $\frac{dB}{dt} \equiv \frac{\Delta B}{\Delta t}$, one obtains $\Delta B \frac{\Delta E}{\Delta t} \approx \Delta t \Delta E \geq \hbar$. But $\Delta t \Delta E \geq \hbar$ is a no sense, since t is a commutative variable !

³²See, e.g., Example 4.16.

independently from the initial energy-mass content. However, this argument does not contradict the Einstein's conservation mass-energy equation ! In fact, the concept of "quantum-time" is only an apparent time, different from the proper time of the quantum relativistic frame, and existing with respect to the interaction quantum-system-quantum-relativistic-frame. Instead the proper time of the observer, belongs to \mathbb{R} , hence commutes with the hamiltonian. In other words with respect to this proper-time, the Heisenberg's uncertainty principle does not apply. \square

Definition 4.24 (Apparent quantum observed time in quantum reaction). *We call (observed) apparent quantum time in a quantum reaction, $\sigma^2(\hat{t})$ entering in equation (48).*

Example 4.25 (Quantum time and strengths of the fundamental interactions). *An experimental way to use the concept of quantum time and apparent quantum time, is to estimate the type of interaction present in. In fact, by using the Heisenberg uncertainty relation $\Delta \hat{t} \Delta E \geq \hbar$, applied to the quantum time and the strengths of the interactions, it is possible to guess the type of interaction from the observed interaction time. In Tab. 6 such a correlation is reported. There are also reported Planck's constants and quarks masses for any related convenience.*

• *For example, if one observes a decay into an apparent quantum time $\sigma^2(\hat{t}) \approx 10^{-23} s$, one can state that it is a strong decay (or quantum-gravity decay) there.*

TABLE 6. Quantum time and strengths of the fundamental interactions.

Apparent interaction time	Interaction type
$10^{-10} s$	Weak
$10^{-16} s$	Electromagnetic
$10^{-23} s$	Strong (quantum-gravity)

$$\hbar = 6.584 \times 10^{-25} GeV s.$$

Comparison with Planck's constants, electron, proton and quark mass-energies									
Planck length	$l_{pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} m$	Quarks	Mass-energy	charge	S'	C'	B'	T'	
Planck mass	$m_{pl} = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} Kg$	up (u)	0.3	$+\frac{2}{3}$	0	0	0	0	
Planck energy	$E_{pl} = m_{pl} c^2 = c^2 \sqrt{\frac{\hbar c}{G}} = 1.22 \times 10^{19} GeV$	down (d)	0.3	$-\frac{1}{3}$	0	0	0	0	
Planck time	$t_{pl} = \frac{l_{pl}}{c} = \frac{\hbar}{m_{pl} c^2} = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44} s$	charm (c)	1.25	$+\frac{2}{3}$	0	+1	0	0	
Planck charge	$q_{pl} = \sqrt{4\pi\epsilon_0 \hbar c} = 1.875 \times 10^{-18} C$	strange (s)	0.5	$-\frac{1}{3}$	-1	0	0	0	
Planck temperature	$\theta_{pl} = \frac{m_{pl} c^2}{K_B} = \sqrt{\frac{\hbar c^5}{G K_B^2}} = 1.416 \times 10^{38} K$	top (t)	91.00	$+\frac{2}{3}$	0	0	0	+1	
$m_e = 0.51 \times 10^{-3} GeV$	$m_p = 0.938 GeV$	bottom (b)	4.8	$-\frac{1}{3}$	0	0	-1	0	

Baryon number quarks $B = \frac{1}{3}$. Baryon number antiquarks $B = -\frac{1}{3}$. $\{S', C', B', T'\} = \{\text{strangeness, charm, bottomness, topness}\}$.

Mass-energy in GeV. m_e = electron mass-energy. m_p = proton mass-energy.

- *An annihilation quark-antiquark, should be observed into an apparent quantum time $\sigma^2(\hat{t}) \approx 3.61 \times 10^{-27}, 1.097 \times 10^{-24} s$. (Look the quark masses reported in Tab. 6. The first refers to $t + \bar{t}$ and the last to $u + \bar{u}$.)*
- *Similarly an annihilation electro-positron, should be observed into an apparent quantum time $\sigma^2(\hat{t}) \approx 6.582 \times 10^{-22} s$. (Look the electron mass reported in Tab. 6.)*
- In the following we characterize the concept of confinement for quantum systems.

TABLE 7. Phenomenological conservation laws in quantum reactions and symmetric reactions.

Name	Symbol	Conserved	Remark
Baryon number	B	yes	$B = \frac{1}{3}(n_q - n_{\bar{q}}) \equiv \frac{1}{3}\Delta n_q$ (n_q = number of quarks, $n_{\bar{q}}$ = number of antiquarks).
Electric charge	Q	yes	$\frac{Q}{e} = I_3 + \frac{Y}{2}$, (e = electron's electric charge) (See Theorem 4.10.) (Gell-Mann-Nishijima formula). (See Theorem 4.8.)
Color charge		yes	
Energy-mass	$E = mc^2$	yes	(See Theorem 4.23.)
Isospin	I	yes in strong int.	
3-Isospin	I_3	yes in strong int.	
Lepton number	$L = L_e + L_\mu + L_\tau$	yes	$L = n_l - n_{\bar{l}}$ (n_l = number of leptons, $n_{\bar{l}}$ = number of antileptons).
Electronic lepton number	L_e	yes	$L_e = n_{e^- + \nu_e} - n_{e^+ + \bar{\nu}_e}$ ($n_{e^- + \nu_e}$ = number of electrons+electron-neutrinos) ($n_{e^+ + \bar{\nu}_e}$ = number of positrons+electron-antineutrinos).
Muonic lepton number	L_μ	yes	$L_\mu = n_{\mu + \nu_\mu} - n_{\bar{\mu} + \bar{\nu}_\mu}$ ($n_{\mu + \nu_\mu}$ = number of muons+muon-neutrinos) ($n_{\bar{\mu} + \bar{\nu}_\mu}$ = number of antimuons+muon-antineutrinos).
Tauonic lepton number	L_τ	yes	$L_\tau = n_{\tau + \nu_\tau} - n_{\bar{\tau} + \bar{\nu}_\tau}$ ($n_{\tau + \nu_\tau}$ = number of tausons+tau-neutrinos) ($n_{\bar{\tau} + \bar{\nu}_\tau}$ = number of antitauons+tau-antineutrinos).
Strangeness number	S'	yes in strong int.	$\Delta S' = 1$ in weak int.
Hypercharge	Y	yes in strong int.	no in weak int.
Crossing symmetry		yes	$n \rightarrow p + e^- + \bar{\nu}_e \Leftrightarrow \bar{\nu}_e + p \rightarrow n + e^+$ (neutron decay) \Leftrightarrow (neutrino dedection).
CPT symmetry	CPT	yes	$CPT = (\text{charge-conjugation})(\text{parity})(\text{time-reversal})$

Hypercharge: $Y = B + S' + C' + B' + T'$, flavour $\equiv (I_3, S', C', B', T') = (3\text{-isospin, strangeness, charm, bottomness, topness})$.

$I_3 = \frac{1}{2}(\Delta n_u - \Delta n_d)$ = 3th-component of isospin I . $S' = -\Delta n_s$. $C' = \Delta n_c$. $B' = -\Delta n_b$. $T' = \Delta n_t$.

$Y = \frac{1}{3}[\Delta n_u + \Delta n_d - 2(\Delta n_s + \Delta n_b) + 4(\Delta n_c + \Delta n_t)]$, $\Delta n_q \equiv n_q - n_{\bar{q}}$.

For complementary information see Wikipedia - List of particles.

Definition 4.26 ((De)confined quantum system). • A solution V of (\widehat{YM}) encodes a confined system if the spectrum $Sp(H(q))$ of the corresponding quantum Hamiltonian, has non-empty point spectrum: $Sp(H(q))_p \neq \emptyset$, for all $q \in V$. Then we say also that V is a confined quantum solution.

• When a solution V of (\widehat{YM}) has non-empty point spectrum only in some subsets N of V , then we say that V encodes a partially confined system and that N encodes the deconfined subsystem. Then we say also that N is the quantum deconfined part of the solution V .

• Whether a solution V of (\widehat{YM}) is such that the following two conditions are satisfied:

(i) $Sp(H(q))_p \neq \emptyset$, for all $q \in V$;

(ii) $Sp(H(q))_c \neq \emptyset$, for all $q \in V$;

then we say that V encodes a confined system that can be deconfined.

Theorem 4.27 ((De)confinement criterion). Let us define the following set-mapping:

$$(49) \quad \begin{cases} \ker_\Delta : \widehat{A} \rightarrow A \\ \ker_\Delta(\hat{a}) \equiv \bigcup_{\lambda \in Sp(\hat{a})} \ker(\hat{a} - \lambda e). \end{cases}$$

Then a solution V of (\widehat{YM}) encodes a confined system (or a confined system that can be deconfined), iff $\ker_\Delta(H(q)) \supset \{0\} \in A$, $\forall q \in V$.

Furthermore, there exists a quantum deconfined part $N \subset V$ of V , iff there exist points $q \in V$, such that $\ker_\Delta(H(q)) = \{0\} \in A$.

Proof. Let us note that if $\lambda_1 \neq \lambda_2 \in Sp(\hat{a})_p$, then $(\hat{a} - \lambda_i e)$, $i = 1, 2$, is not injective and $\ker(\hat{a} - \lambda_1 e) \cap \ker(\hat{a} - \lambda_2 e) = 0 \in A$. Instead, if $\lambda \in Sp(\hat{a})_c \cup Sp(\hat{a})_r$, then

$(\hat{a} - \lambda e)$ is injective, hence $\ker(\hat{a} - \lambda e) = \{0\} \subset A$. Therefore, in the points $q \in V$ where V encodes a confined quantum system, or a confined quantum system that can be deconfined, then necessarily $\ker_\Delta(H(q)) \supset \{0\} \in A$. Instead, in the points $q \in V$, where V encodes a deconfined quantum system, then necessarily must be $\ker_\Delta(H(q)) = \{0\} \in A$. \square

• Let us conclude this paper by answering to the following question: "Does quantum Majorana neutrino *exist* ?".³³

Definition 4.28 (Quantum Majorana neutrino). *We define quantum Majorana neutrino a quantum massive, electrically neutral fermionic particle, its own antiparticle, identified with a Cauchy data of the quantum super Yang-Mills PDE (\widehat{YM}) .*

Theorem 4.29 (Existence of quantum Majorana neutrino). *The quantum super Yang-Mills PDE (\widehat{YM}) admits quantum nonlinear propagators $\tilde{V} \subset (\widehat{YM})$ such that $\partial\tilde{V} = \tilde{N}_0 \sqcup \tilde{P} \sqcup \tilde{N}_1$, where \tilde{N}_1 can be identified with Majorana neutrino. Furthermore, the following propositions hold.*

- (i) *A complex quantum quasi-particle exists that is a quasi neutralino, and that we call quantum Majorana neutralino. This contains two Majorana neutrinos other than two Higgsinos and two supersymmetric partner of the couple $(\nu_e, \bar{\nu}_e)$.*
- (ii) *\tilde{V} is homeomorphic to a quantum $(4|4)$ -superdisk with two attached superhandles.*

Proof. Let $V \subset (\widehat{YM})$ be a quantum nonlinear propagator in (\widehat{YM}) such that

$$(50) \quad \partial V = N_0 \sqcup P \sqcup N_1$$

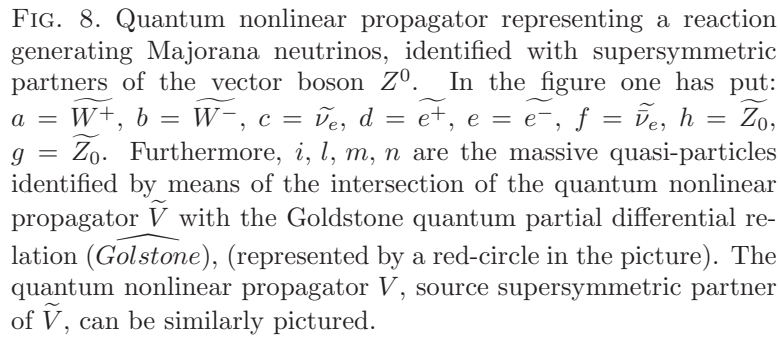
with $N_0 = W^+ \sqcup W^-$ and $N_1 = Z^0 \sqcup Z^0$. This is possible since V represents the reaction $W^+ + W^- \rightarrow Z^0 + Z^0$ that splits in the intermediate decays $W^+ \rightarrow \nu_e + e^+$, $W^- \rightarrow \bar{\nu}_e + e^-$ and reactions $e^+ + e^- \rightarrow Z^0$, $\nu_e + \bar{\nu}_e \rightarrow Z^0$. A representation by means of elementary bordisms is given in Fig. 8. Let us, now, consider the following lemmas.

Lemma 4.30 (Boson and Fermion quantum super PDEs). *In the quantum super Yang-Mills PDE, (\widehat{YM}) there exist two disjoint, open PDEs $(\widehat{Boson}) \subset (\widehat{YM})$ and $(\widehat{Fermion}) \subset (\widehat{YM})$, that we call respectively Boson-PDE and Fermion-PDE, that are formally integrable and completely integrable. Solutions of (\widehat{Boson}) , (resp. $(\widehat{Fermion})$), are boson-polarized, (resp. fermion-polarized).*

Proof. With respect to the definitions in Tab. 5, we can write $\hat{A} = \mathfrak{b} \sqcup \mathfrak{f} \sqcup \mathfrak{n}$, where $\mathfrak{n} \equiv \hat{A} \setminus \mathfrak{b} \sqcup \mathfrak{f}$.³⁴ We identify such disjoint subsets as equivalence classes of an equivalence relation \mathcal{R} in \hat{A} and endow the quotient \hat{A}/\mathcal{R} with the quotient topology,

³³This question arises from the paper *Symmetrical theory of electron and positron*, published by E. Majorana [43], about a solution of the Dirac equation, where he first showed existence of a massive, electric-neutral, spin $\frac{1}{2}$, solution, that coincides with its anti-particle. This solution is now usually called *Majorana neutrino*. Existence of this particle should allow the so-called *neutrinoless double beta decay* of some nuclei. (See, e.g., this beautiful link <http://thy.phy.bnl.gov/vo-gelsan/GGS/Wilkerson.pdf>.)

³⁴Let us emphasize that \mathfrak{b} is not empty since $\hat{a} \equiv h^2 s(s+1) id_A$, $s = n \in \{0, 1, 2, \dots\}$, belong to \mathfrak{b} . Similarly \mathfrak{f} is not empty since $\hat{a} \equiv h^2 s(s+1) id_A$, $s = \frac{2n+1}{2}$, $n \in \{0, 1, 2, \dots\}$, belong to \mathfrak{f} .



Lemma 4.31 (Supersymmetric partners of particles). *Cauchy data $N_0 \subset (\widehat{Boson})$, (resp. $N_1 \subset (\widehat{Fermion})$), are called bosonic particles, (resp. fermionic particles). All the bosonic particles (resp. fermionic particles), belonging to the same integral bordism class in (\widehat{Boson}) , (resp. $(\widehat{Fermion})$), are considered dynamically equivalent. If there exists a quantum nonlinear bordism V of (YM) , such that $\partial V = N_0 \sqcup P \sqcup N_1$, where N_0 is a bosonic particle and N_1 is a fermionic particle, and P is an integral quantum supermanifold, such that $\partial P = \partial N_0 \sqcup \partial N_1$, then we say that (N_0, N_1) is a couple of supersymmetric partners. The following propositions are equivalent.*

Proof. The equivalence of propositions (i) and (ii) follows from Theorem 3.6 in part I and Theorem 4.10 in [79](II). \square

Lemma 4.32 (Supersymmetric partners quantum nonlinear propagators). *Let N_0 and N_1 be respectively initial and final Cauchy data in a quantum reaction in (YM) , such that $N_0 \equiv b_1 \sqcup \cdots \sqcup b_r \sqcup f_1 \sqcup \cdots \sqcup f_s$ and $N_1 \equiv \bar{b}_1 \sqcup \cdots \sqcup \bar{b}_{\bar{r}} \sqcup \bar{f}_1 \sqcup \cdots \sqcup \bar{f}_{\bar{s}}$, with b_i , $i = 1, \dots, r$, and $\bar{b}_{\bar{i}}$, $\bar{i} = 1, \dots, \bar{r}$, bosonic, and f_j , $j = 1, \dots, s$, $\bar{f}_{\bar{j}}$, $\bar{j} = 1, \dots, \bar{s}$, fermionic. Let V be a quantum nonlinear propagator of (YM) such that $\partial V = N_0 \sqcup P \sqcup N_1$, and $\partial P = \partial N_0 \sqcup \partial N_1$. Then there exists a quantum nonlinear propagator \tilde{V} of (YM) such that $\partial \tilde{V} = \tilde{N}_0 \sqcup \tilde{P} \sqcup \tilde{N}_1$, and $\partial \tilde{P} = \partial \tilde{N}_0 \sqcup \partial \tilde{N}_1$. We call \tilde{V} the quantum supersymmetric partner of V . Therefore to any quantum reaction there corresponds a quantum supersymmetric partner reaction.*

Proof. This follows from Lemma 4.31 and from the commutative diagram (51), holding for any quantum conservation law α of (YM) .

$$(51) \quad \begin{array}{ccc} \langle \alpha, N_0 \rangle + \langle \alpha, N_1 \rangle & \xlongequal{\quad} & - \langle \alpha, P \rangle \\ \parallel & & \parallel \\ \langle \alpha, \tilde{N}_0 \rangle + \langle \alpha, \tilde{N}_1 \rangle & \xlongequal{\quad} & - \langle \alpha, \tilde{P} \rangle \end{array}$$

□

Now from Lemma 4.32 applied to quantum nonlinear propagator (50), we see that its supersymmetric partner \tilde{V} , represents the following reaction $\tilde{W}^+ + \tilde{W}^- \rightarrow \tilde{Z}^0 + \tilde{Z}^0$. Here \tilde{Z}^0 is just a massive neutral fermionic particle, its own antiparticle. Thus \tilde{Z}^0 can be identified with a Majorana neutrino.³⁵ Therefore, we can conclude that Majorana neutrinos can be identified with possible products of quantum reactions in (YM) , and the proof of the first part of the theorem is down. In order to prove (i), let us emphasize that the quantum nonlinear propagator \tilde{V} , identifies also a complex quasi-particle, that we call *quantum Majorana-neutralino*, $\nu_{Maj-neutralino} \equiv (g, h, i, l, m, n)$. Here i and m , are the supersymmetric partners of massive neutrinos, ν_m , $\bar{\nu}_m$, identified in Theorem 4.18, and m and n , are supersymmetric partners of Higgs quasi particles, (say *Higgsinos*), according to Theorem 4.13 and Theorem 4.15. In other words the quantum super-Majorana-neutrino is a quasi *neutralino*. This last has two photinos (the supersymmetric partner of photons) instead of the couple $(\tilde{\nu}_e, \tilde{\bar{\nu}}_e)$. So it appears that the name is justified ! In order to complete the proof, i.e., to prove proposition (ii), let us remark that $\partial \tilde{V}$ belongs to the same singular integral bordism class of a quantum exotic homotopy (3|3)-supersphere, $\hat{\Sigma}^{3|3}$, since $\Omega_{3|3,s}^{(YM)} = 0$. However, \tilde{V} is not homeomorphic to $\hat{D}^{4|4}$, but to a (4|4)-superdisk with two attached super-handles. This means that the corresponding observed solution by means of a quantum relativistic frame $i : N \rightarrow M$, is necessarily a singular solution of $(YM)[i]$.³⁶ □

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³⁵In the particle-zoo this is usually called *zino*.

³⁶Let us emphasize that the quantum nonlinear propagators V and \tilde{V} , considered in this proof are not elementary ones in the sense of Theorem 3.2. In fact V and \tilde{V} are not homeomorphic to $\hat{D}^{4|4}$.

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